COMBINATORICS

~ 13 students → pigeons

months → holes

January, Feb... Decem

13 → pigeon

colors → holes

12

✓

26 1-letter

× 26×26 2-letter

+ 26×26×26 3-letters

+ 26^4 4 letters

5,6,0,000

$$26 + 26^2 + 26^3 + 26^4$$

$$= \underbrace{4,75,254}_{< 50}$$

$\underline{26} \times \underline{26} \times \underline{26}$

$\underline{a}$ $\underline{b}$

a a

a a

a b

⋮

a z

b a

b b

⋮

b z

c a

c b

⋮

c z

⋮

---

13 → 3 × 4 = "12" → holes

3 first names

4 last names

Seeta Raju

Geeta Raju.

Seeta Raju

Seeta Ramam

k_1

k_2

k_i

k_j

k_N

$$\boxed{N = 100}$$

$N-1$

$$\boxed{99}$$

$$k_1 = q_1 \cdot 99 + \underbrace{r_1}$$

$$0 \leq \underbrace{r_1} \leq 98$$

$$\underbrace{\checkmark 0, \checkmark 1, \checkmark 2, \checkmark 3, \ldots , 98 \checkmark}_{\text{"holes"}}$$

$$k_i = q_i \, 99 \checkmark + \boxed{r_i} \checkmark$$

$$\boxed{k_j} = q_j \, 99 \checkmark + \boxed{r_j} \checkmark$$

$$\underbrace{N-1}$$

N

$$N' < \boxed{N}$$

$$r_i = r_j .$$

3
7
10
101

3
5
7
11
13
17

$[n] = \{1, 2, 3, \dots n\}$

$\boxed{[200]} = \{1, 2, \dots, 200\}$

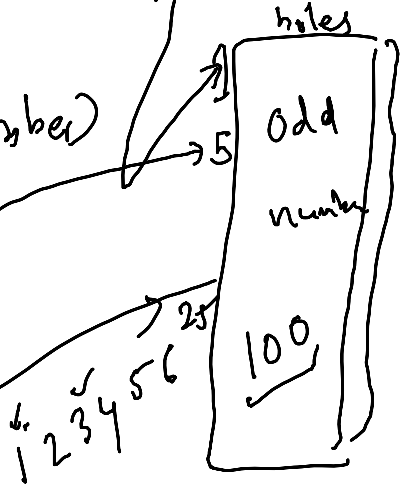
$P_1 \quad P_2$

$36 = 2^5 \cdot 1$ 101 numbers → pigeons

$K \to 2^i \cdot (\text{odd number})$

$10 \to 2^1 \cdot \underline{5}$

$100 = 2^2 \cdot 25$



$$k = 2^{i} \cdot \underbrace{\text{"odd}_1\text{"}}_{\cdot} \checkmark$$

$$= 2^{j} \text{"odd}_2\text{"}$$

$$99 = 2^{0} \cdot \textcircled{99} \checkmark$$

$$a, b \quad \textcircled{\text{"101"} \checkmark} \checkmark$$

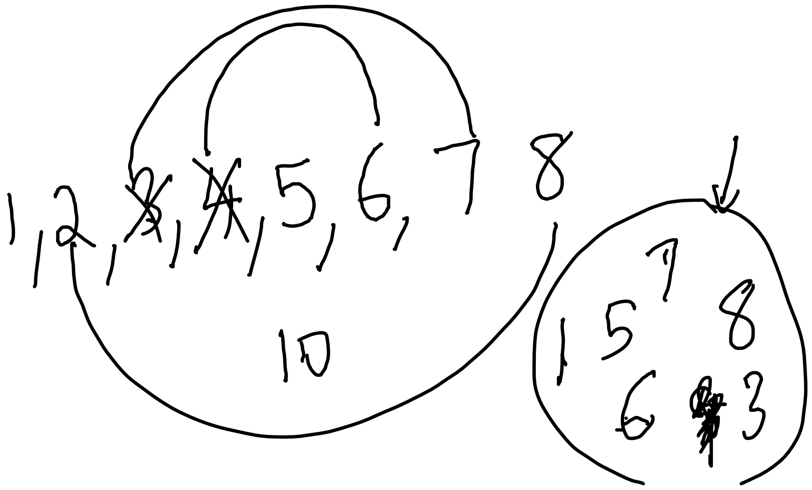
number

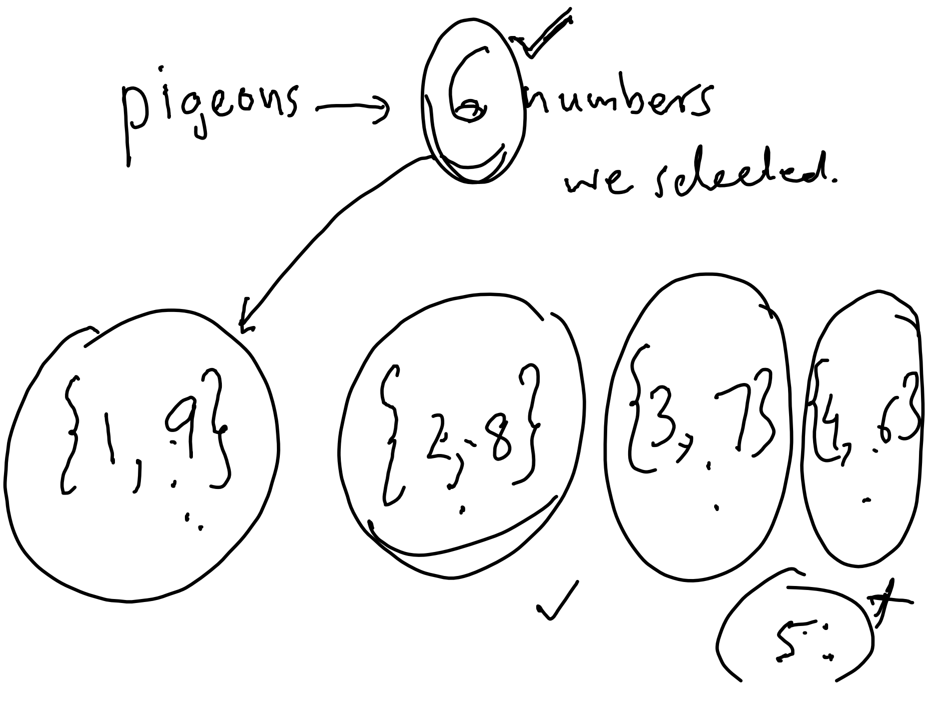
$$a = 2^{i} \cdot x \checkmark \qquad b > a$$

$$b = 2^{j} \cdot x \checkmark \qquad a, b \in \boxed{x}$$

$$\boxed{j > i} \qquad \underline{x \cdot 2^{i}} \,\Big|\, \underline{x \cdot 2^{j}}$$

$\{1, 2, \ldots 9\} = [9]$


1, 2, 3, 4, 5, 6, 7 8
10
1 5 7 8
6 3


pigeons → 6 numbers
we selected.
{1, 9}
{2, 8}
{3, 7}
{4, 6}
5
✓

$|S| = 6$

$S = \{ \overset{1}{n_1}, n_2, n_3, \dots, n_6 \}$

$n_1 + n_2 + \dots + n_6$

14+
13
12+
11
10+
9

$n_1 < n_2 < n_3 < \dots < n_6$

$64 = 2^6$ S

A 010110

$n_6 \le 14$

$2^6 - 1 = 65$

pigeons

$S = \{ \underbrace{n_1, n_2, \dots, n_6} \}$

$n_1 < n_2 < \dots < \underline{n_6 \le 14}$

$2^6 - 1 = \underline{63}$

9

$A \subseteq S$ sum(A)

$1 \le n_1$ 1, 2, ..., $\sum_{i=1}^{6} n_i$ $= \sum_{n_i \in A} n_i$

$\underbrace{S=\{n_1, n_2, \ldots, n_6\}}$ ✓

$$\left[\begin{array}{ll} n_6 & 14+ \\ & 13 \\ \vdots & 12 \\ n_3 & 11 \\ n_2 & 10 \\ n_1 \to & \cancel{9} \end{array}\right]$$ ✓

$\boxed{69 - 9} = \underline{60}$

1 to 69

$\boxed{69}$ holes

$\boxed{63}$ pigeons

holes $= \underline{60}$

$63 - 1 = \underbrace{62}$

1 hole

m an odd +ve integer

$\underbrace{2^n - 1}$ m = "7"

$2^1 - 1 = 1, \; 2^2 - 1 = 3, \; \underbrace{2^3 - 1 = 7}$

"5"

$2^1-\cancel{1=1}$, $2^2-\cancel{1}=3$, $2^3-\cancel{1=7}$,

$2^4-1=15$

m

2^1-1, 2^2-1, 2^3-1, ... $\boxed{2^{m+1}-1}$

2^m-1

$\boxed{m+1}$ ← pigeons

$\overbrace{\cancel{0}, 1, 2, 3, \ldots m-1}=m$ "m" – holes

$$\underbrace{1 \le i < j \le m+1}$$

$$2^i - 1 = q_1 m + r$$

$$2^j - 1 = q_2 m + r$$

$$\overbrace{\cancel{2^i}\left(2^{j-i} - 1\right)} = \underbrace{2^j - 2^i} = (q_2 - q_1)\boxed{m} \leftarrow p_1^{k_1}$$

$$\boxed{m} \mid 2^{j-i} - 1$$

$$\boxed{n = j-i}$$

7, 77, 777, 7777, . . .

2003

2003

$\underbrace{1, 2, \cdots 2002}$

$777777\cdots\cdots7 \leftarrow q_1 \cdot 2003 + r$

$= j$ digits

$7777\cdots7 \leftarrow q_2\ 2003 + r$

i digit

$$\underbrace{7777}_{j-i}\ \underbrace{000\ 0\ 00}$$

$\leftarrow$ divisible by $\underbrace{2003}$

$(7777\cdots7 \times 10^{i})$

2003

$1 \quad 2 \quad 3 \; \cdots \; \overbrace{i \;\; i+1 \;\; i+2} \qquad 28$

$a_1 \quad a_2 \quad a_3 \qquad a_i \; a_{i+1} \qquad a_{28}$

$1 \quad 2 \quad 3 \quad 4 \quad 1$ "15"

$x_1 < x_2 < \underbrace{\overset{\downarrow}{x_j} \cdots *} < x_{28}$

$x_1 = a_1$

$x_2 = a_1 + a_2$ ✓

$x_3 = a_1 + a_2 + a_3$ ✓

$\vdots$

$x_{28} = a_1 + \cdots + a_{28}$ ✓

x_1
$\wedge$
x_2
$\wedge$
x_3
$\wedge$
$\vdots$
x_{28}

$$28 \quad \underbrace{\overset{1}{x_1},\ x_2,\ x_j}\,, \cdots \underbrace{x_i}\ \ \underbrace{x_{28}} \leftarrow 40$$

$$28 \quad \underset{14}{x_1+15},\ \underset{14}{x_2+15},\ \cdots,\ \underset{14}{x_{28}+15}$$

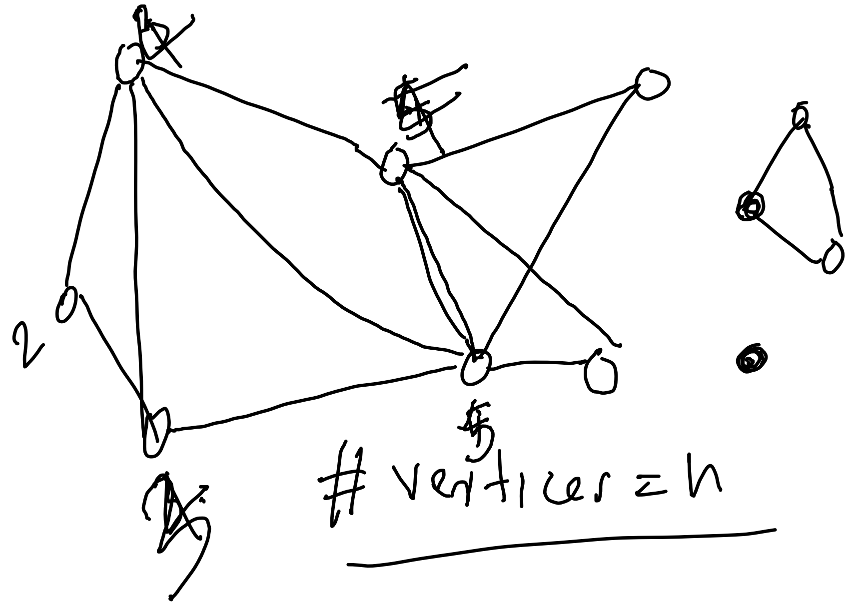
$$\overline{56} \qquad \underbrace{1, \cdots, 55} \qquad \downarrow$$

$$40+15$$

$$=\underbrace{55}_{\boxed{54}}$$

$$x_i = x_j + 15 \qquad \underbrace{i > j}$$

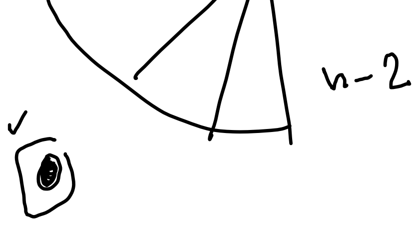
$$\underbrace{x_i - x_j} = \underbrace{15}$$



vertices = n

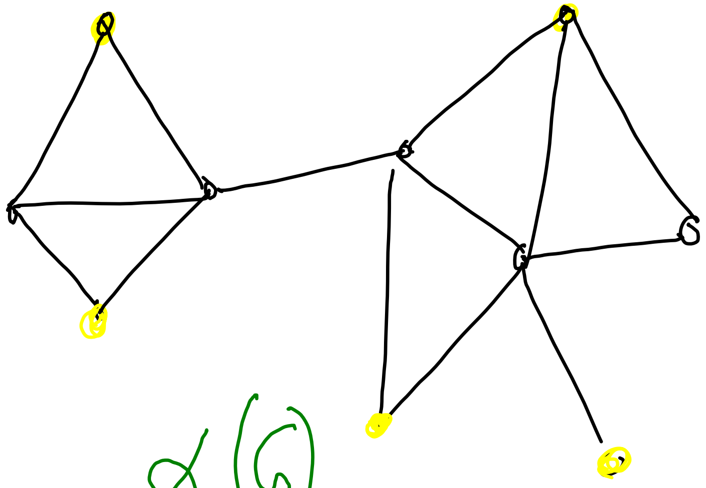
n

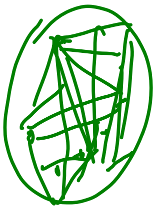
$\cancel{0},\ \underbrace{1, 2, \dots \overset{i}{\boxed{\ }} \dots \cancel{n-1}}$



n−1 ← holes

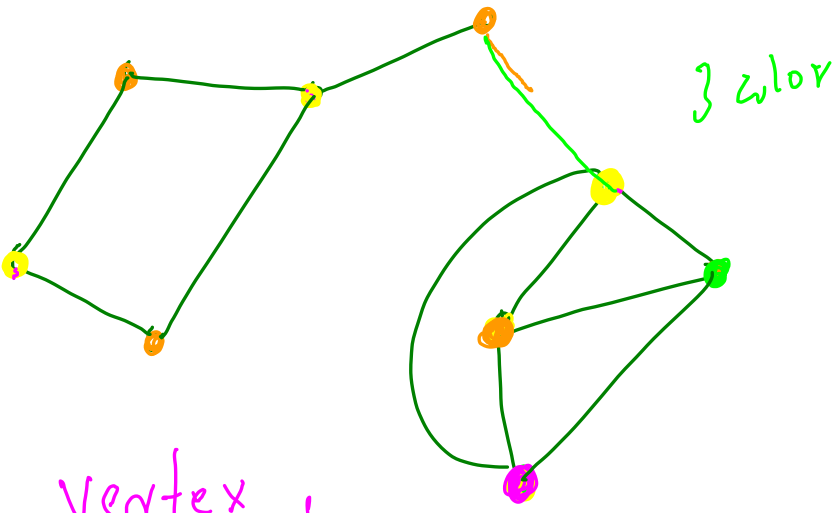
n vertices





$\alpha(G)$

— stability number



$\chi(G)$ vertex coloring — proper

$$\boxed{n \geq \chi(G)\cdot\alpha(G)}$$

color classes → holes

vertices → pigeons

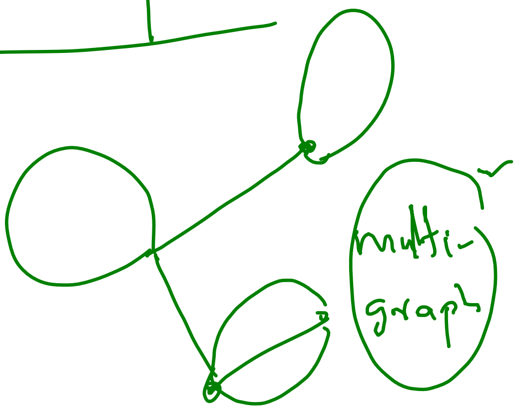
$\chi(G) \longleftarrow n \quad \frac{n}{\chi(G)}$

$n = p_1^{k_1} p_2^{k_2} p_3^{k_1} \cdots$

Grimaldis
Ramana

— Diestel
— Bondy & Murty
— Harary

Finite, simple



$\alpha(G)$ $\chi(G)$

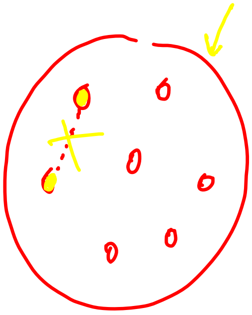
Stabilitynumber

maximum independent set

n — vertices → pigeons

$\chi(G)$ colors → "hole"

n



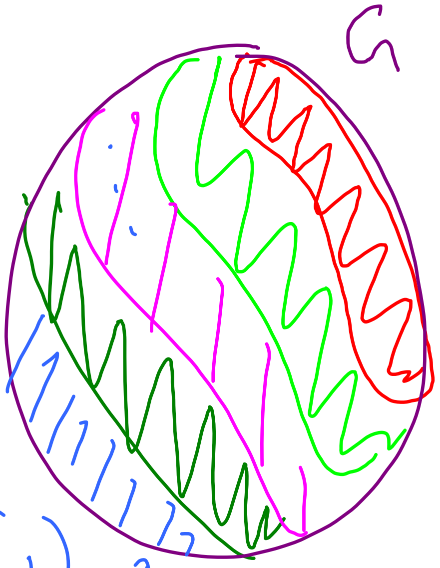
co

$$\frac{n}{\chi(G)}$$

$$\alpha(G) \geq \left\lceil \frac{n}{\chi} \right\rceil$$

$$n > \chi \cdot k \quad \leq \left(\frac{n}{\chi} - 1\right)$$

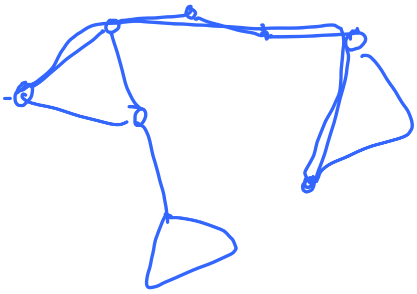
$$(k+1)$$

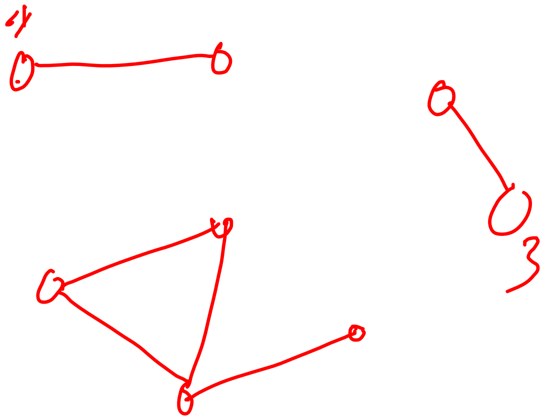


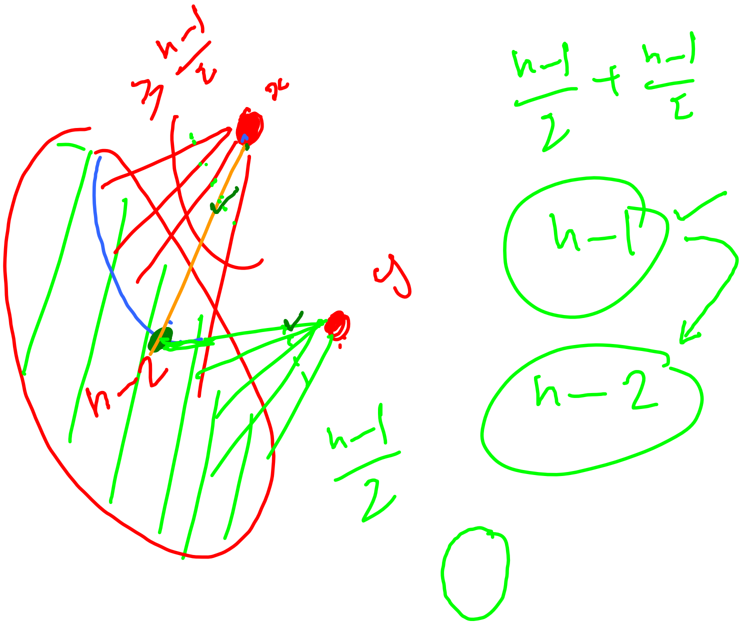
G

$n = |G|$

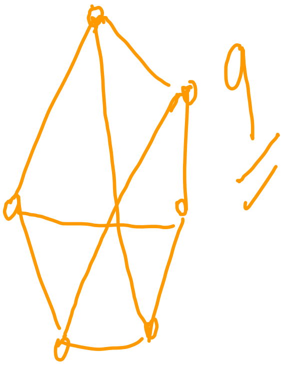
mindegree $\geq \frac{n-1}{2}$




4
3

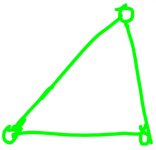

≥ (n-1)/2
x
y
n-2
(n-1)/2
(n-1)/2 + (n-1)/2
n-1
n-2

G



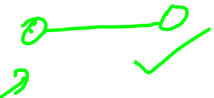
2n=6 $n^2+1=9+1=10$

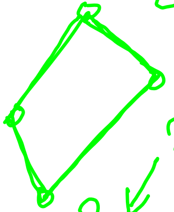
n=3



2n

2n=2

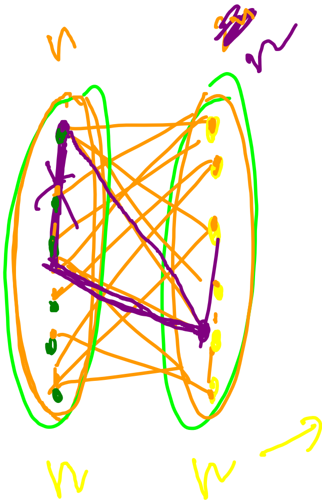




n^2+1

$2^2+1=5$

2n=4



Complete
bipartite
graph

$n \times n = n^2+1$

$(n+1)^2 + 1$ edges

$2(n+1)$

$2 \cdot 1 = 2$

n

$2 \cdot 2 = 4$

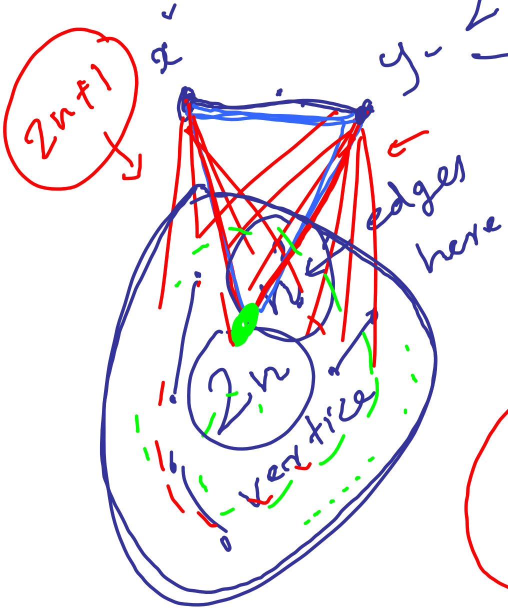
$2n$

$\boxed{2 \cdot 3 = 6}$ ✓

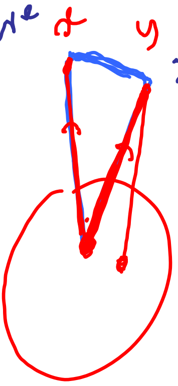
$\boxed{n+1}$ ✓

n^2+1 $\underline{2n}$

$\underline{2n+2} = 2(n+1)$

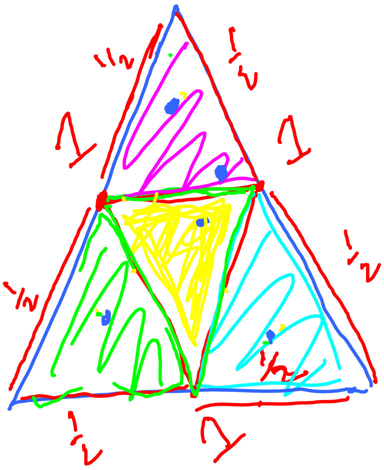


$(n+1)^2 + 1$



$$\underline{= n^2 + (2n+1) + \boxed{1}}$$

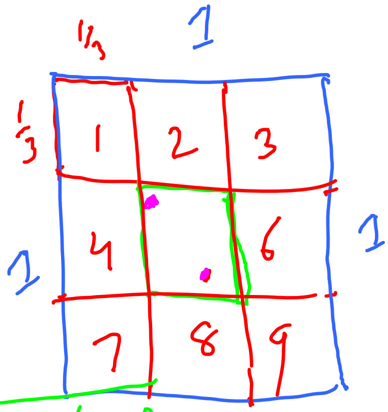
edges



pigeons

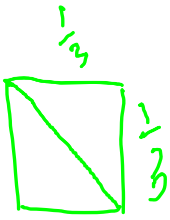
5 points

$> \frac{1}{2}$



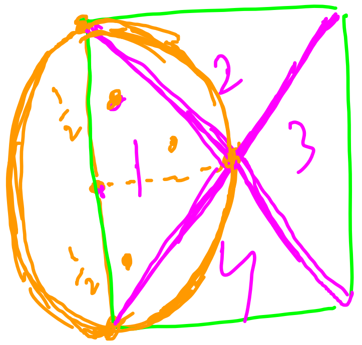
10 pts

< 0.48cm



$$\sqrt{\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2} = \frac{\sqrt{2}}{3} = 0.471.. < 0.48$$

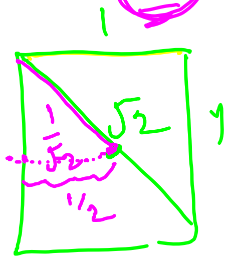
10



$10 > 4 \cdot 2 = 8$

$\boxed{3}$

$\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$



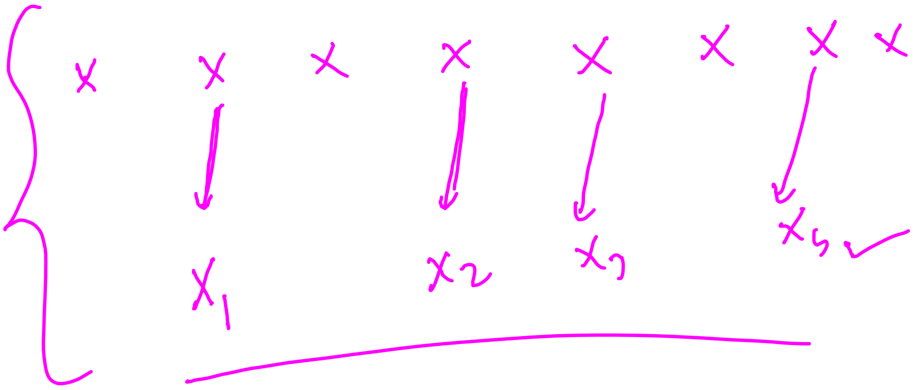
$\boxed{a_2}$ a_1 a_3
a_2 a_3 a_1
a_1 a_2 a_3
a_3 a_2 a_1
a_3 a_1 a_2
a_1 a_3 a_2

$a_1, a_2, \ldots, a_{n^2+1}$

distinct red numbers

$(n^2+1)!$ $\quad 3! = \underline{3 \times 2 \times 1}$

n^2+1



$n=1$

$n^2+1=2$ 1

$\underline{10, \quad 20}$ $\underbrace{n+1=2}$

$20, 10$ $n=2$

$n^2+1=5$ $\boxed{n+1}$

1 16 12 9 , 7 ✓

"3" $n+1$ "2"$+1$

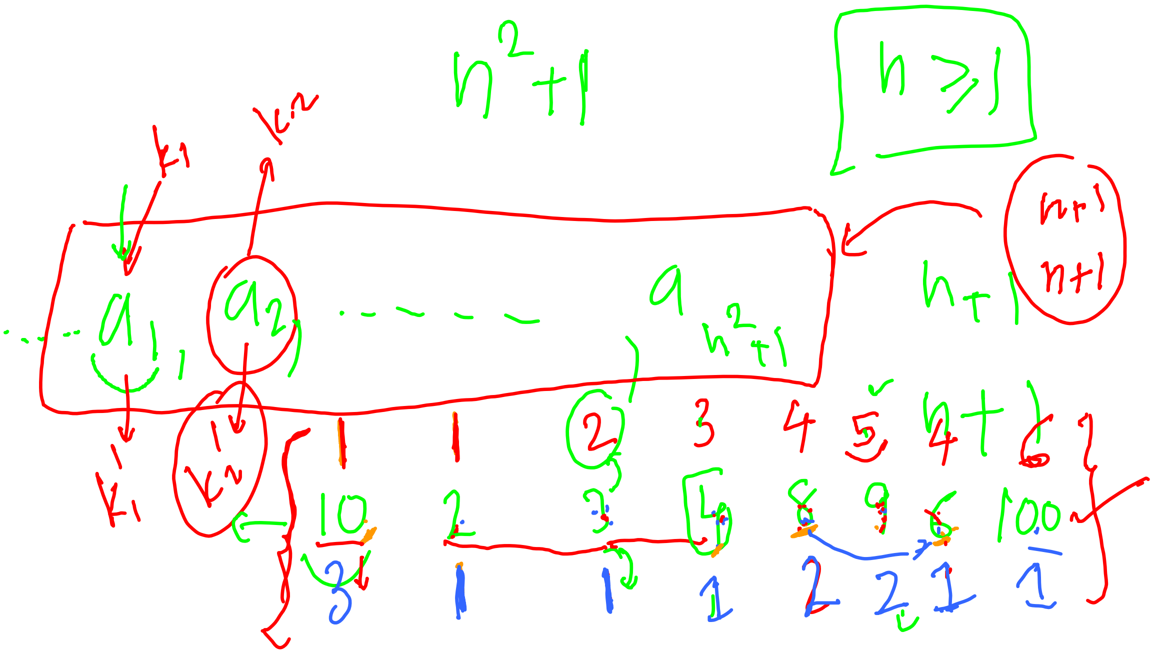
$=3$

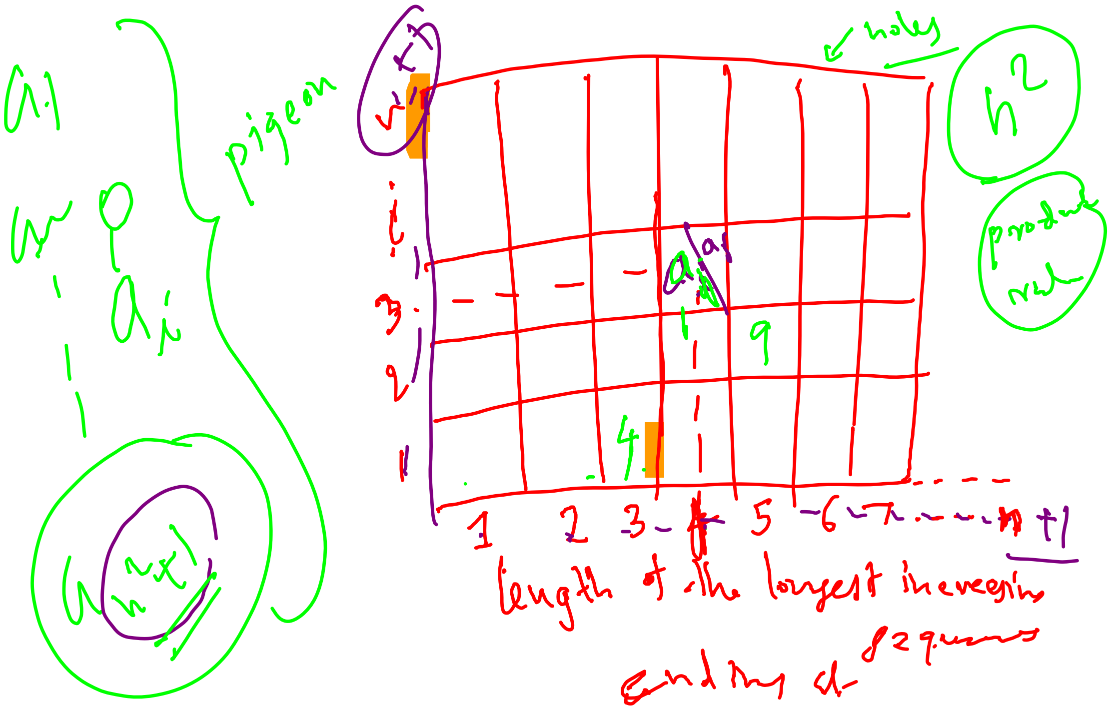
$100 \times 100+1$ $n=100$

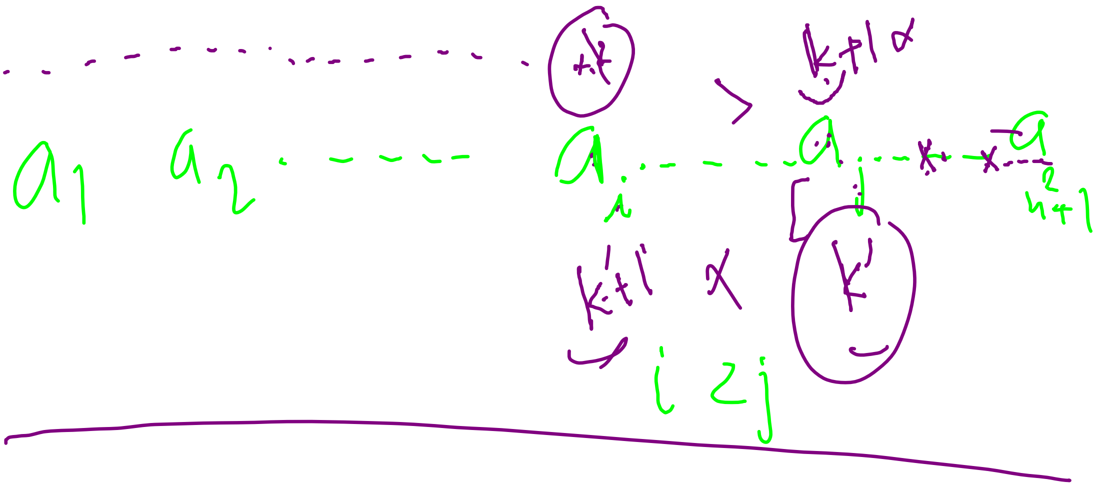
101 $\boxed{101}$ $\boxed{100}$ ✓

100

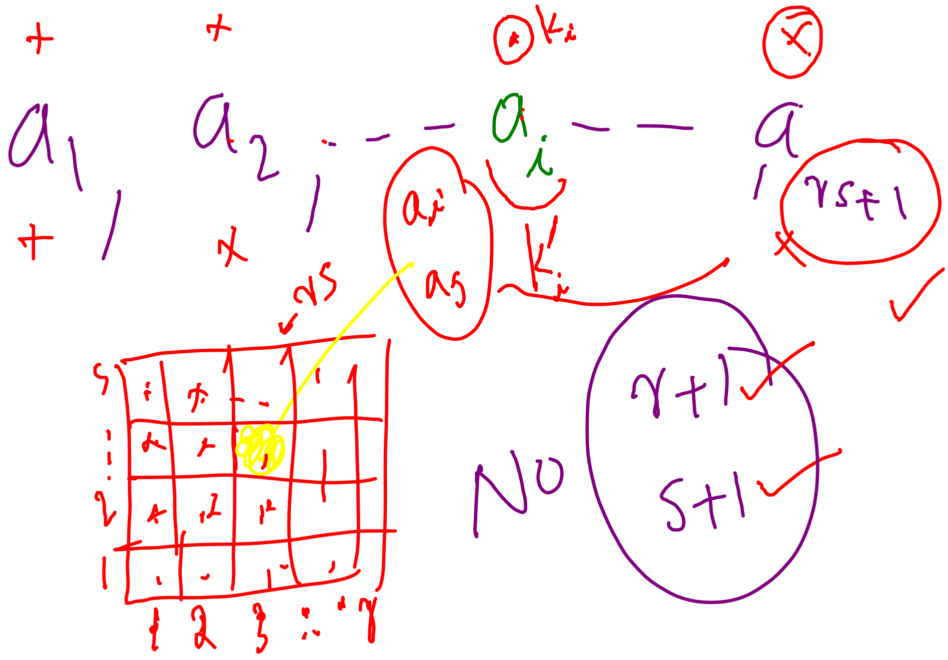
"101"


n^2+1
n ≥ 1
k1
k2
a1, a2 ...... a n^2+1
n+1
n+1
n+1
1 1 2 3 4 5 4+1 6
10 2 3 4 8 9 6 100
3 1 1 1 2 2 1 1

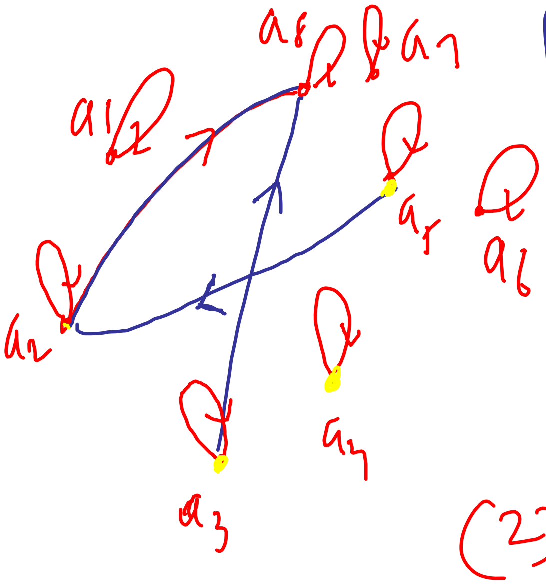

a1
ai
a n^2+1
pigeon
n+1
holes
n^2
product rule
1 2 3 4 5 6 7 ..... n+1
1 2 3 ... n+1
ai
aj
4
9
length of the longest increasing
ending at squares



k
k+1 α
>
a1
a2
ai
aj
a n+1
2
i
j
k+1
x
k
i 2 j



+
+
ki
x
a1
a2
ai
a
i
+
x
ai
as
rs
k'i
rs+1
x
NO
r+1
s+1
s
2
1
1
2
3
r



$(a_8, a_2) \in R$

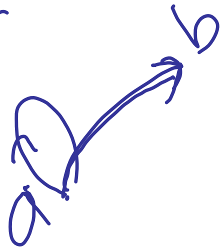
$(a_2, a_8) \in R$

(1) Reflexive

$(a,a) \in R$

(2) ~~anti-~~ symmetric

transitive

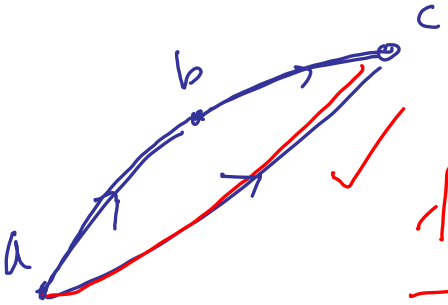


$(a,b) \in R$

$(b,a) \in R$

$(a,b) \in R \quad (b,c) \in R \quad a=b$

$\Rightarrow (a,c) \in R$



transitive

$a < b \mid b < c$

$\Rightarrow a < c$

R

b

$(i, j) \in R$

$a,$

$a \leq b$

$a < b$

$b < a$

$i \leq j$

$\{1, 2, 3, 4, - - — \}$

$(a, b) \in R$

$(5, 10) \in R$

$(2, 88) \in R$ $\quad a \mid b$

$$\boxed{\begin{array}{l}(3, 2) \notin R \\ (2, 3) \notin R\end{array}}$$

$$\left.\begin{array}{l}a \mid a \\ a \mid b \\ b \nmid a\end{array}\right\}$$

$a \mid b \quad b \mid c \Rightarrow a \mid c$

A, B

$[n] = \underbrace{\{1, 2, 3, \dots n\}}_{[n]}$

$A \subseteq B \subseteq C \quad 2^n$

ϕ

$A \subseteq C$

R

$\{1, 2\}$

$\{2, 3\}$

$(A, B) \in R$

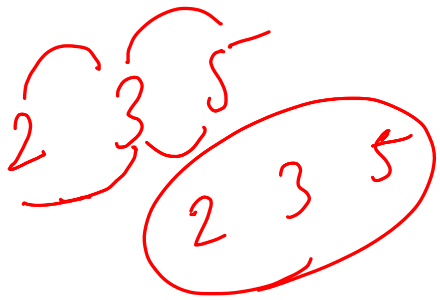
$A \subseteq B$

$1 \le 2 \le 10 \le 18 \le 20$

$5, 10, 100, 200, 1000$

$\underline{a_1 < a_2 < a_3 < \dots a_k}$

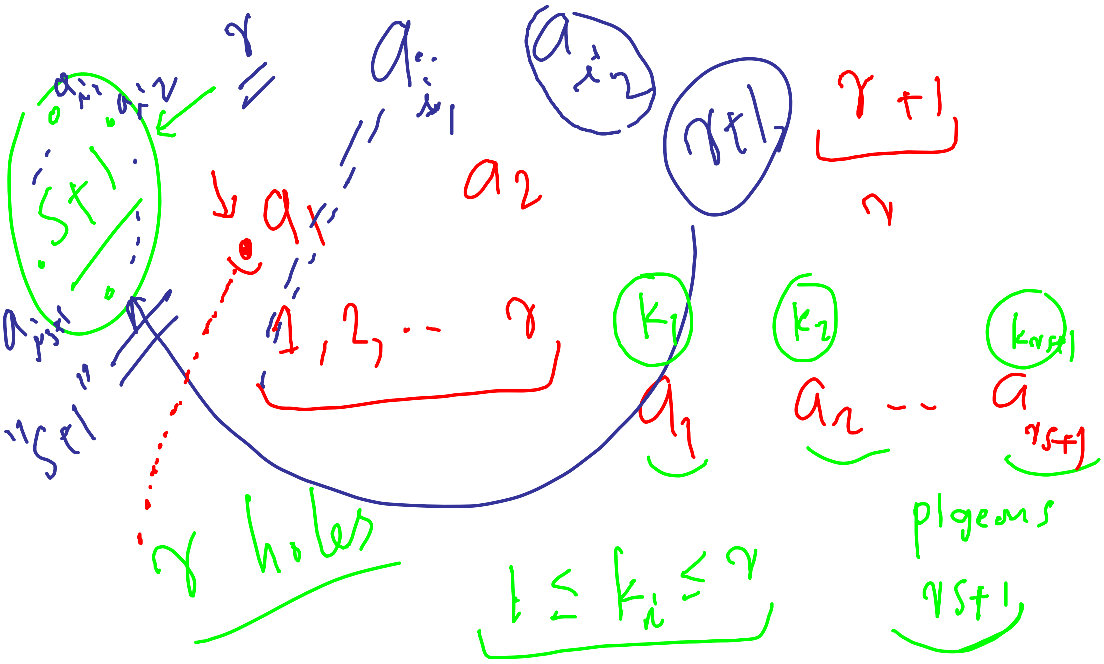
$a_1 \quad a_2 \quad a_3$



$$|P| \geq rs+1$$

either $r+1$ "r"

$s+1$



5 digit passwords

$8 \times 10 \times 10 \times 10 \times 10$

$\times 8$

$8 \times 10^4 +$

$8 \times 10^4 +$

$10 \times$

$\overset{\checkmark}{8}\ \overset{\checkmark}{8}\ \underline{6}\ \underline{5}\ \underline{4}$

$\underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }\ \underline{\ \ }$

$9 \times 10 \times 10 \times 10 \times 10$

9×10^4 _

8?

$\underset{\_}{\ }\ \underset{\_}{9}\ \underset{\_}{9?}\ \underset{\_}{9}\ \underset{\_}{9}$

8×9^4

1, 2

2, 1

$2! = 2 \times 1$

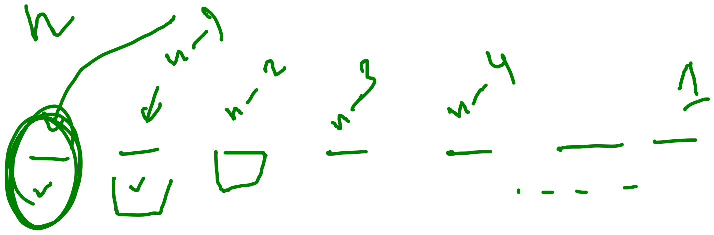
1, 2, 3 2 1 3 3 1 2

1, 3, 2 2 3 1 3 2 1

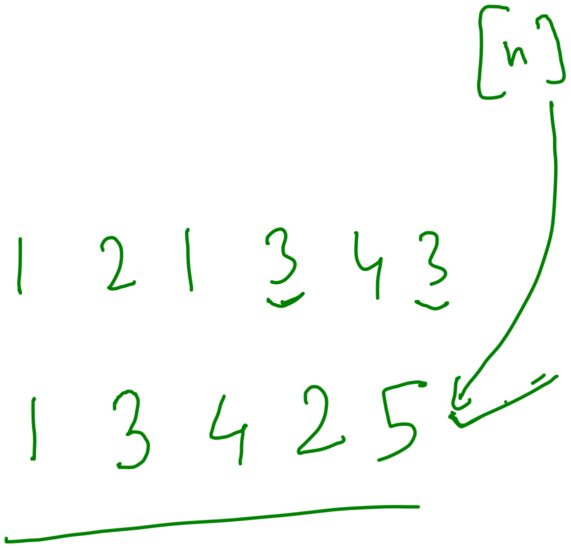
6 =

$3 \times 2 \times 1$

$\{1, \cancel{2}, \cancel{\ldots}, n\}$



$$n \times (n-1) \times (n-2) \times \cdots \times 1$$
$$= n!$$



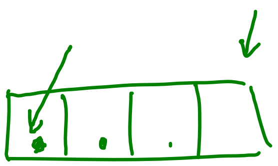
$[n] = \{1, 2, \ldots n\}$ $\quad$ $5P_3 = (5)_3 = \underline{5 \times 4 \times 3}$

$n!$

$\{1, 2, 3, 4, 5\}$ $\quad$ $(n)_k$

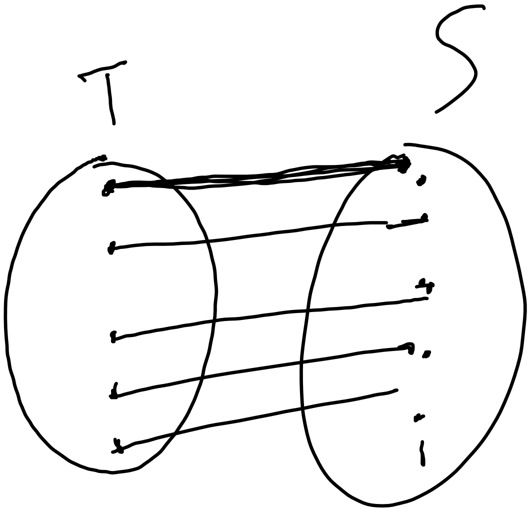
$n = 5$ $\quad$ $k = 3$ $\quad$ $\underbrace{{}_nP_k}$

1 2 3	1 5 4	2 1 3
1 3 2	1 2 4	2 1 4
1 4 5	1 4 2 . . .	2 1 5

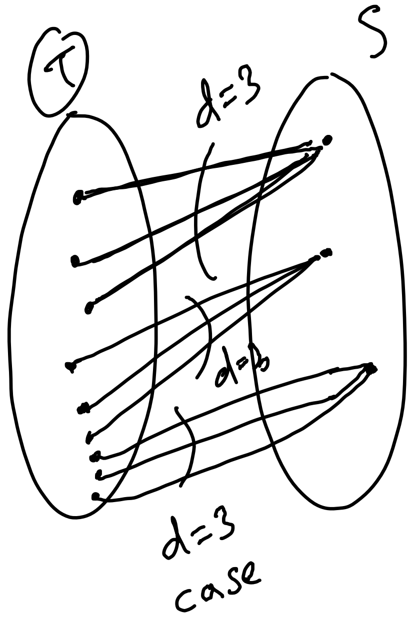


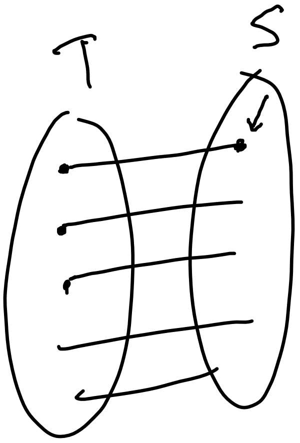
$\overset{5}{n} \cdot \overset{4}{(n-1)} \cdot \overset{3}{(n-2)}$

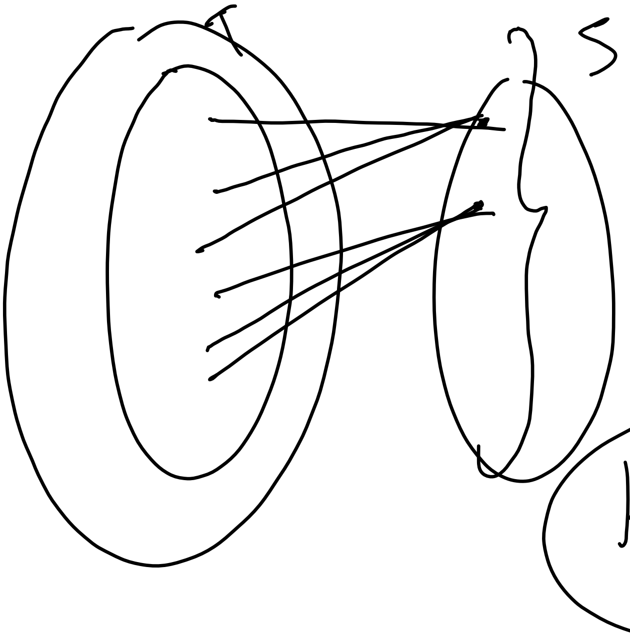
$\cdots \underline{(n-k+1)}$



d-to one fns.



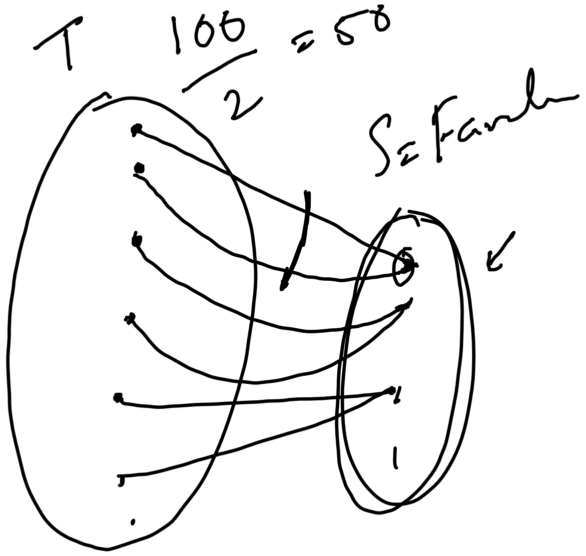


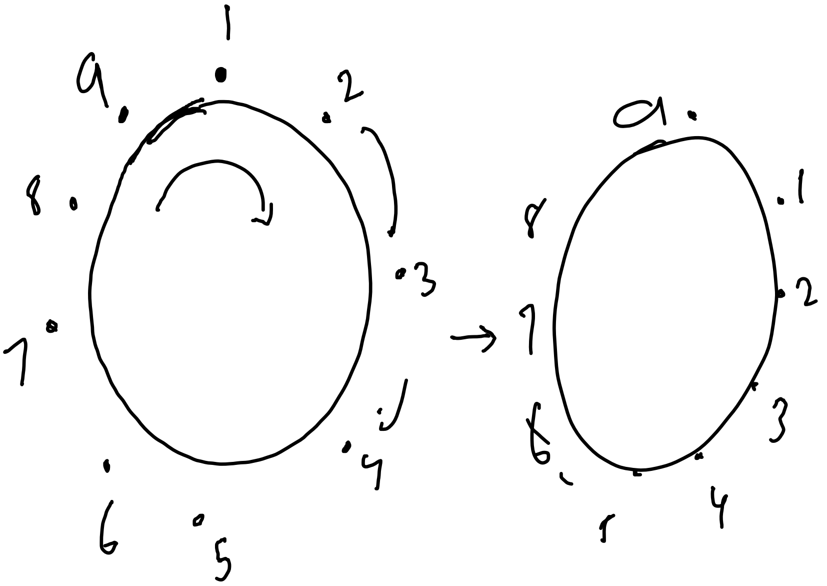


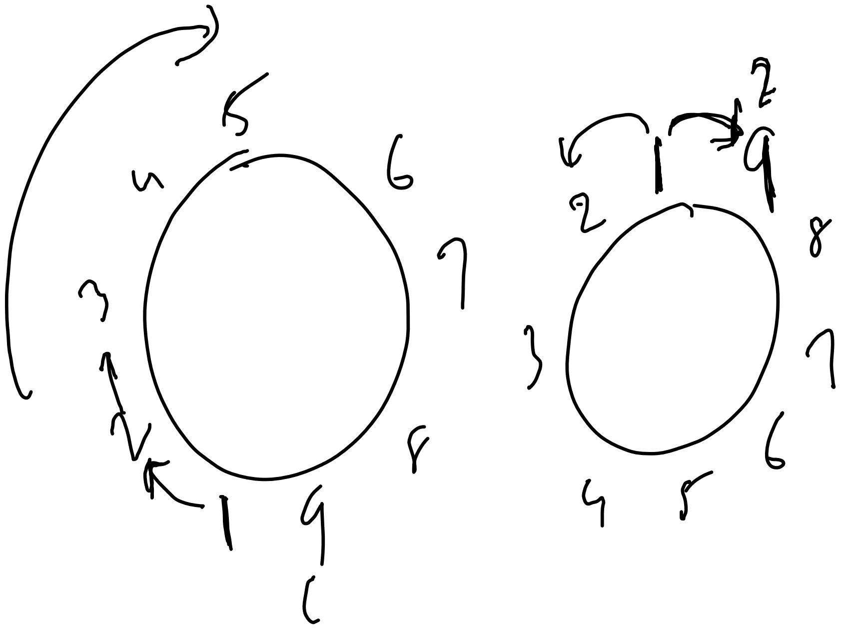
$|S| = k$

$|T| = k \cdot d$

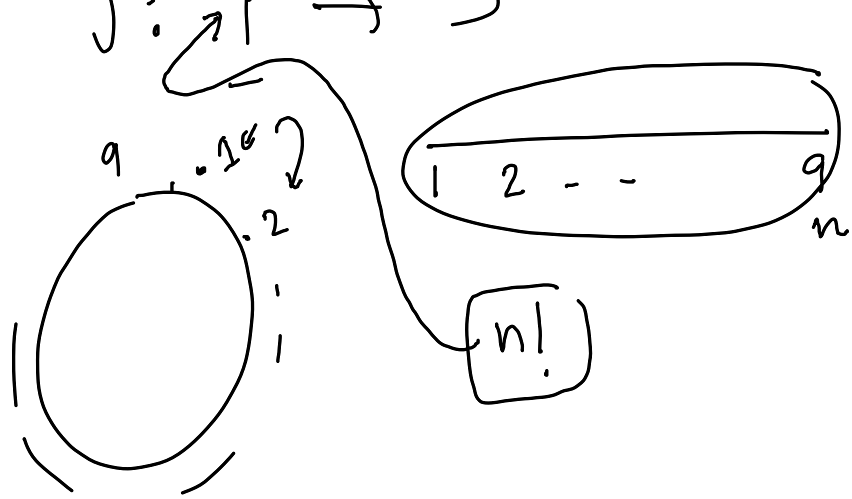
$\boxed{k = \frac{|T|}{d}}$

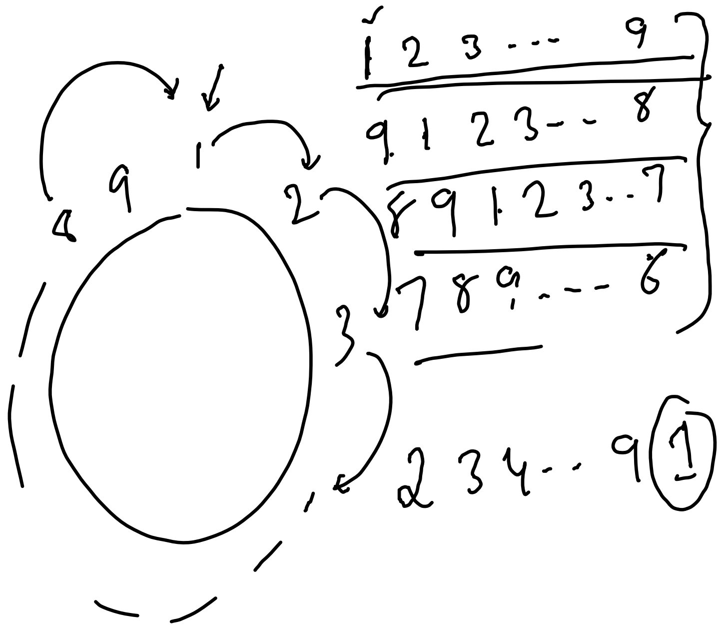

T
100/2 = 50
S=Famh
1


1
a
2
8
3
7
4
6
5
a
8
1
2
7
3
6
4
5

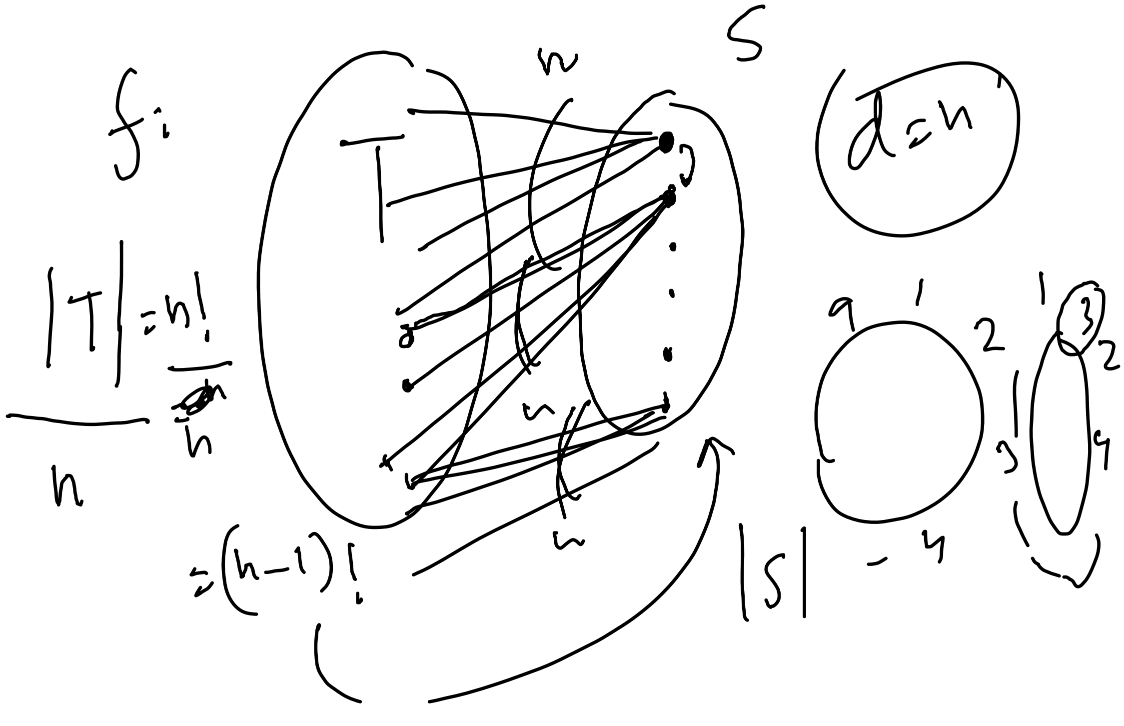


$f : T \to S$





1 2 3 ... 9
9 1 2 3 ... 8
8 9 1 2 3 .. 7
7 8 9 ... 6
2 3 4 .. 9 1
8
9
1
2
3



f:
T
S
n
|T| = n!
n
n
n
=(n-1)!
n
n
d=n
9
1
2
1
3
2
3
4
|S| = 4

$\binom{n}{k}$

$[n]$

k-element

$(1,2,3,4,5)$

$\underbrace{(n)_k}$

$\boxed{{}_nC_k}$

$\underbrace{{}_nP_k}$

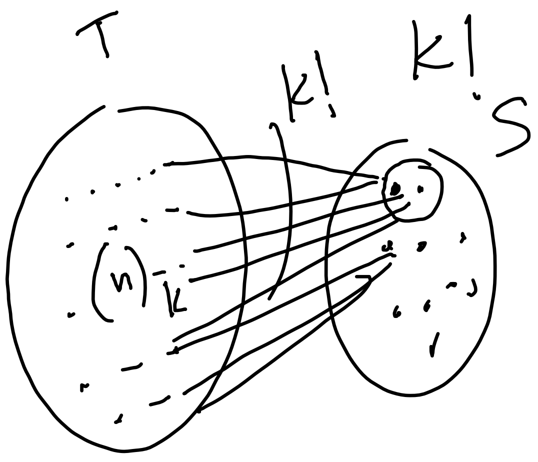
"3"

$\{1,3,2\}$ ✓

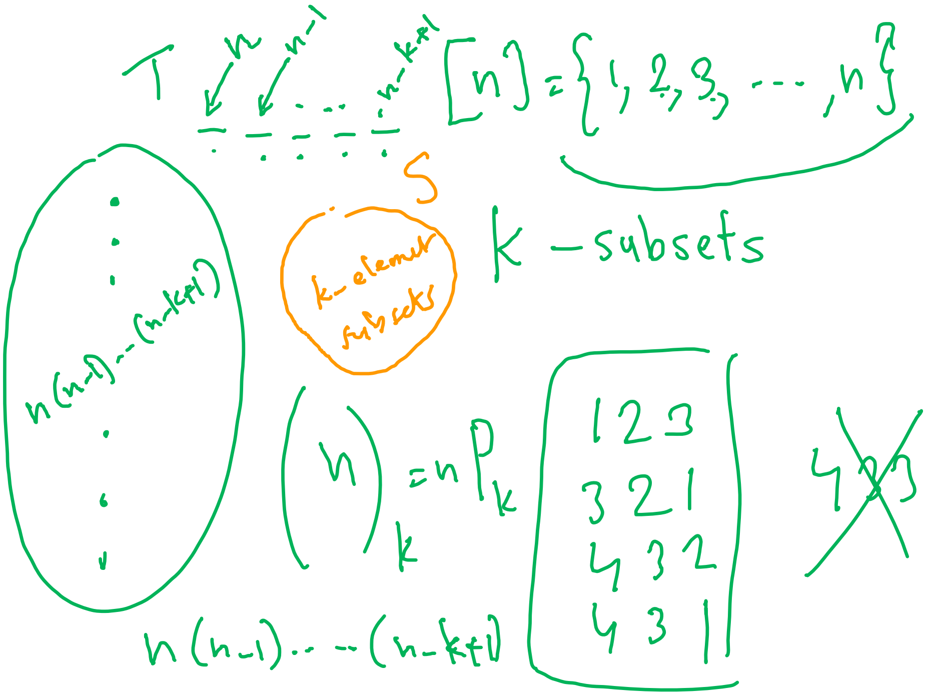
1 2 3 ✓ 1 4 3

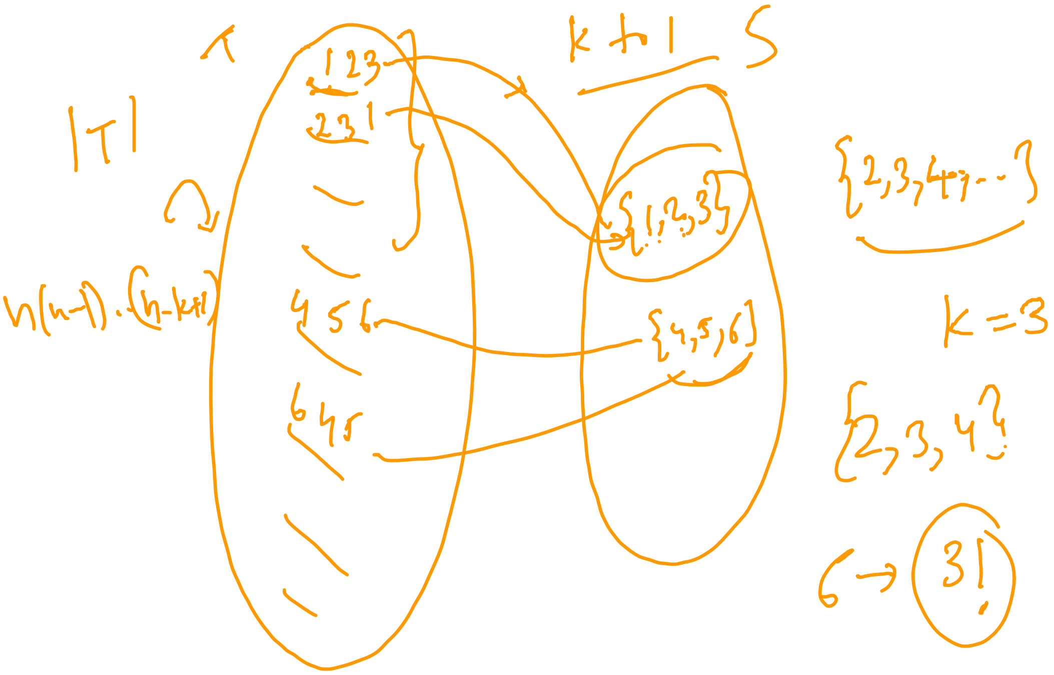
1 3 2 ✓ 1 3 4

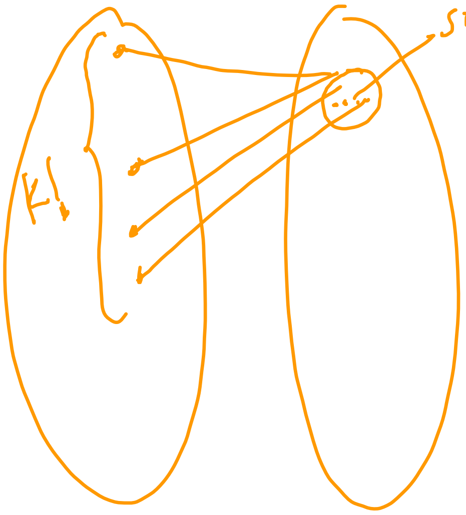
$$\frac{n(n-1)\cdots(n-k+1)}{}$$



$\{1\ 2\ 3,$


T
n
n-1
n-k+1
$[n] = \{1, 2, 3, \dots, n\}$
S
k-element subsets
k-subsets
$n(n-1)\dots(n-k+1)$
$\binom{n}{k} = {}^nP_k$
123
321
432
431
~~432~~
$n(n-1)\dots\dots(n-k+1)$


T
|T|
$n(n-1).(n-k+1)$
123
231
456
645
k to 1
S
$\{1,2,3\}$
$\{4,5,6\}$
$\{2,3,4,\dots\}$
$k=3$
$\{2,3,4\}$
$6 \to 3!$



subset of cardinality k

$$|S| = \frac{\boxed{|T|}}{k!}$$

$$\frac{(n)_k}{k!} = \frac{n(n-1)\dots(n-k+1)}{k!}$$

$${}^{n}C_{k} \qquad \begin{pmatrix} n \\ k \end{pmatrix} \qquad \frac{n!}{n!} = 1$$

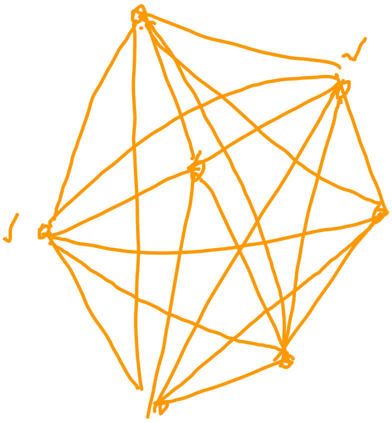
$$\boxed{\underbrace{{}^{n}C_{0}} = 1} \checkmark \qquad \underline{\underbrace{{}^{n}C_{n}} = 1}$$

$$\rightarrow \boxed{0! = 1} \checkmark \qquad \underbrace{n! = n(n-1)\dots}$$

$$\boxed{{}^nC_1 = n}$$

$${}^nC_2 = \frac{n(n-1)}{2} \qquad \{1\}, \{2\}, \ldots \{n\}$$

n vertices.



How many edges in K_n ?

$${}^nC_2 = \frac{n(n-1)}{2}$$

$$(x+y)^n = \binom{n}{0} x^0 y^n +$$

$$\binom{n}{1} x^1 y^{n-1} + \binom{n}{2} x^2 y^{n-2} + \dots$$

$$\dots \binom{n}{i} x^i y^{n-i} + \dots + \binom{n}{n} x^n y^0$$

$$(x+y)^n = \underbrace{(x+y)\,(x+y)\,(x+y) \dots (x+y)}_{n \text{ times}}$$

$$(x+y)^2 = (x+y)\,(x+y)$$

$$x^2 y^0 + \overbrace{\widetilde{x}\,\widetilde{y} + \widetilde{y}\,x}^{2xy} + y^2$$

$$\binom{2}{2} x^2 y^0 + \binom{2}{1} x^1 y + \binom{2}{0} x^0 y^2$$

1 term 2nd term 3rd term

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

$$\binom{3}{0} x^0 y^3 + \binom{3}{1} x^1 y^2 + \binom{3}{2} x^2 y^1 + \binom{3}{3} x^3 y^0$$

1st 2nd nth

$$(x+y)^n = \underbrace{(x+y)(x+y)(\cdots)\cdots(x+y)}_{n \text{ times}}$$

$$\binom{n}{0} x^0 y^n + \binom{n}{1} x^1 y^{n-1} + \binom{n}{2} x^2 y^{n-2} + \cdots + \binom{n}{i} x^i y^{n-i} + \cdots + \binom{n}{n} x^n y^0$$

$$2^n \qquad (\overset{1}{x}+\overset{1}{y})^n = \binom{n}{0}\overset{1}{x^0y^n} + \dots + \binom{n}{i}\overset{1}{x^iy^{n-i}} + \dots + \binom{n}{n}\overset{1}{x^ny^0}$$

$$\boxed{(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}}$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$$

$$\boxed{2^n = \sum_{i=0}^{n} \binom{n}{i}}$$

$$\underbrace{2^n} \quad \left[\begin{array}{l} \binom{n}{0} \longrightarrow \\ + \\ \binom{n}{1} \longrightarrow \\ \vdots \\ \binom{n}{n} \longrightarrow \end{array}\right.$$

counts the # subsets of n with 0 element

$$(\overset{-1}{x} + \overset{+1}{y})^n = \sum_{i=0}^{n} \binom{n}{i} \underset{(-1)^i}{\underset{\downarrow}{x^i}} \overset{1}{\boxed{y^{n-i}}}$$

$$0 = 0^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i$$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + \binom{n}{i} + \cdots + \binom{n}{n}(-1)^n$$

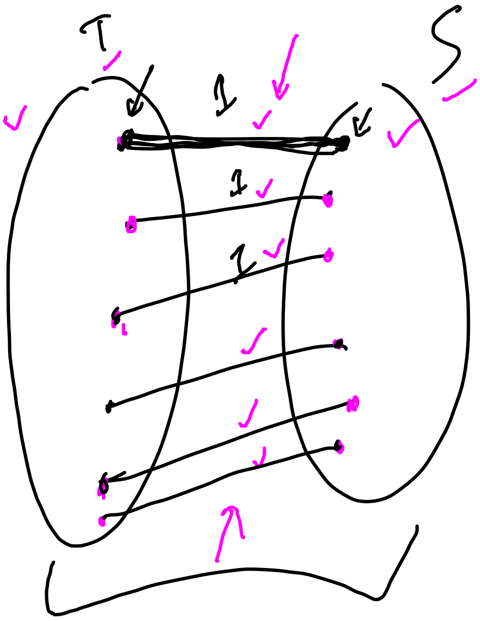
odd

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots$$

$$3^n = (\overset{2}{\overset{\downarrow}{x}} + \overset{1}{\overset{\downarrow}{y}})^n = \sum_{i=0}^{n}\binom{n}{i}2^i$$

$$= \underbrace{\binom{n}{0}2^0 + \binom{n}{1}2^1 + \dots + \binom{n}{n}2^n}$$

$|T| = |S|$



$\underline{d=1}$

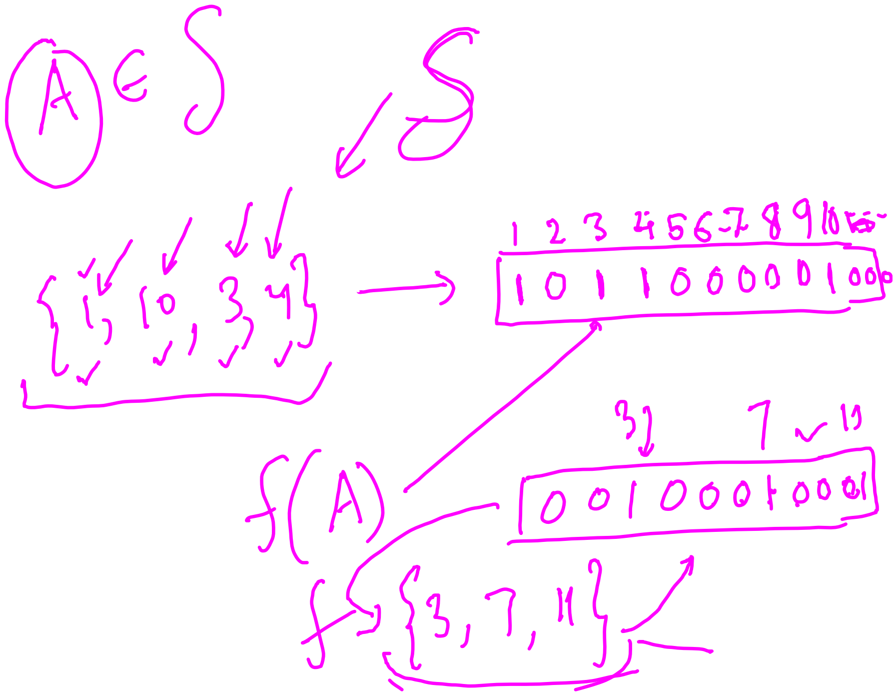
bijection

$$\underbrace{\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}}_{(1+1)^n = 2^n}$$

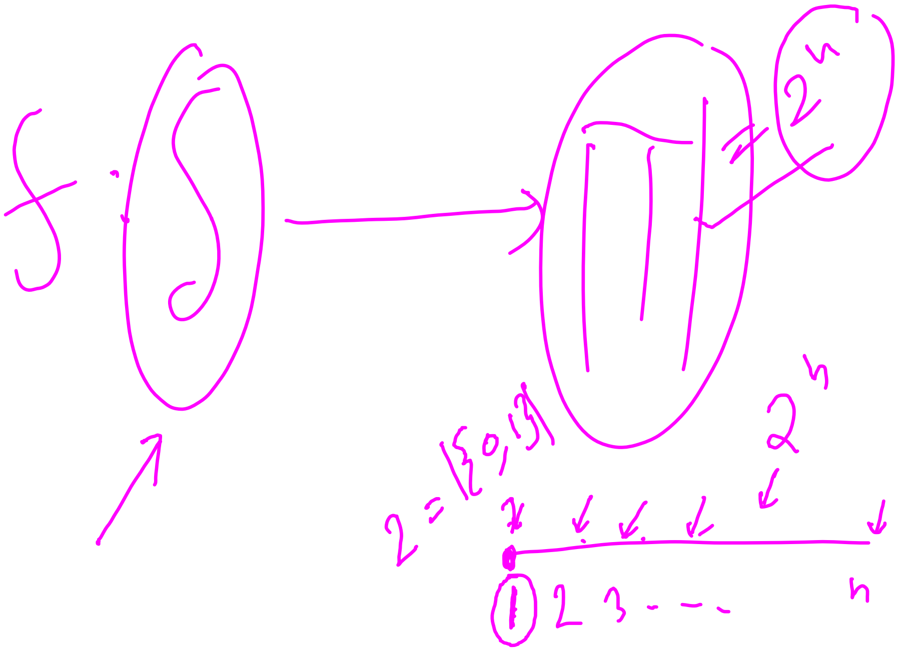
$$\begin{tikzcd} S \arrow[d] \arrow[rd, shift left, "f"] & \\ \text{subsets of } [n] & T \arrow[lu, shift left] \end{tikzcd}$$

$[n] = \{1, 2, 3, \dots n\}$

$T =$ set of $\left(\text{binary strings of length "n".}\right)$



A ∈ S
S
{1, 10, 3, 4}
1 2 3 4 5 6 7 8 9 10
1 0 1 1 0 0 0 0 0 1 0 0 0
f(A)
3 7 11
0 0 1 0 0 0 1 0 0 0 1
f{3, 7, 11}



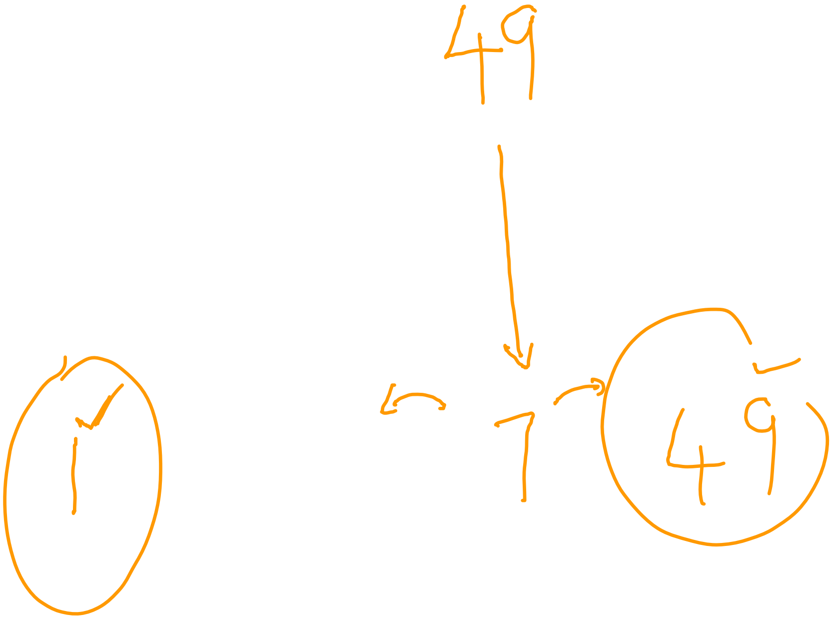
f: S
T
2^n
2 = |{0, 1}|
2^n
1 2 3 - - -
n

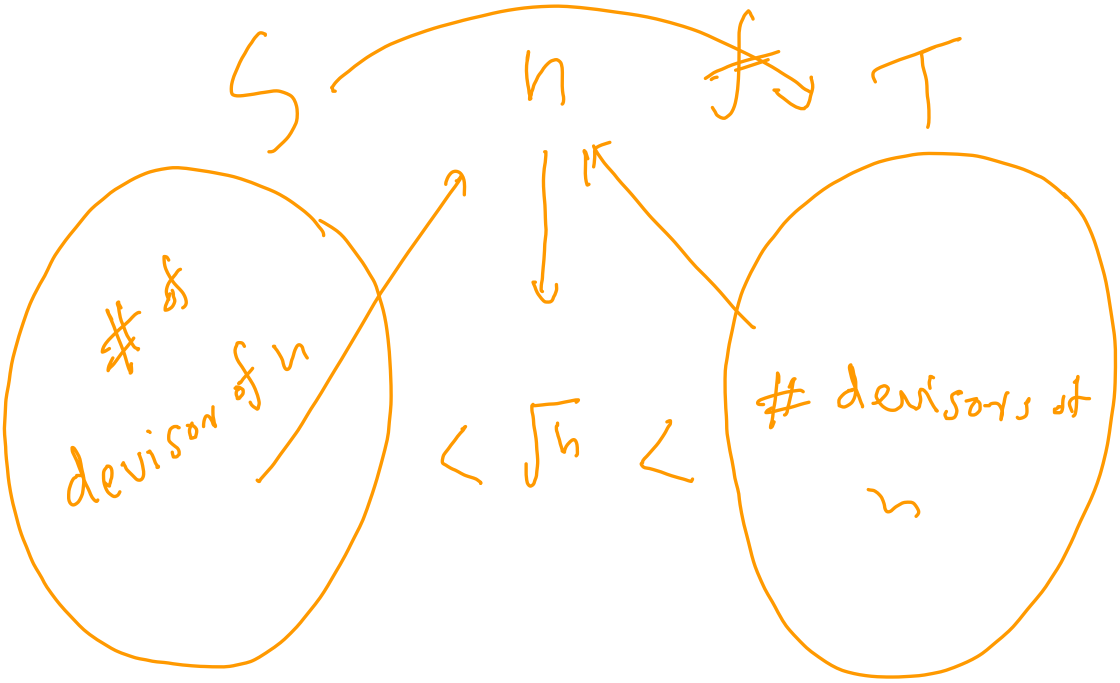
$$\begin{tikzcd} & 10 \arrow[d] \\ & \sqrt{10} = 3 + \varepsilon \\ \underbrace{\{1, 2\}} & \underbrace{\{5, 10\}} \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} 36 \arrow[d] \\ 6 \end{tikzcd}$$

$$\underbrace{[1, 2, 3, 4]} < 6 < \underbrace{[\check{36}, \check{12}, \check{18}, \check{6}]}$$

$$\longleftarrow \sqrt{n} \longrightarrow$$


49
1
7
49


S
n
f
T
# of devisor of n
< √n <
devisors of n

$f : S \longrightarrow T$

$1 \qquad n$

$2 \quad x \quad n/2 = n$

$\vdots$

$d \qquad n/d$

24

$3 \times 8 = 24$

$4 \times 6 = 24$

$1 \;—\; 24$

$2 \; \overset{x}{—} \; 12 = 24$

$d \mid 24$

$\frac{24}{d} \mid 24$

$d \mid n \Rightarrow \frac{n}{d} \mid n$

$f(\boxed{d}) = \boxed{\frac{n}{d}} \in T$

$\frac{n}{d} > \sqrt{n}$

$\downarrow$

$d < \sqrt{n}$

$d \in S$, where S is the set of devisors of n, $< \sqrt{n}$

T — devisors of n $> \sqrt{n}$

$\downarrow$

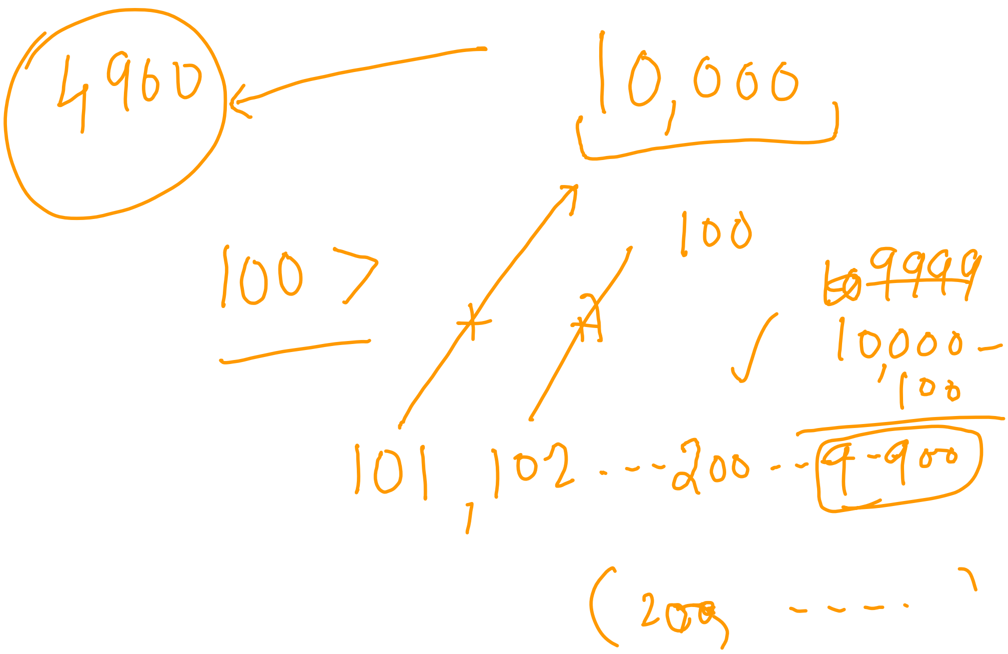
$\sqrt{n} < k$

~~n~~ $\frac{n}{t} = k$

$\boxed{t} = \boxed{\frac{n}{k}} \leftarrow$

$$f : S \to T$$

$$|S| = |T|$$



1 ✓ 2 ✓ ~~3~~ 4 ✓ 99

99 checks

4900/9900 ten

10^8

"n"

100000000

$\sqrt{n}$

10,000

~~2~~ ~~2~~

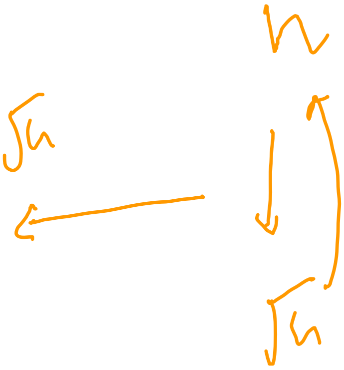
$\sqrt{n}$

$n - \sqrt{n}$

for $k = \sqrt{n}+1$ to n

do . . .

$k = 1$ to $\lfloor\sqrt{n}\rfloor$ -a



What kind of numbers have odd number of divisors?

6

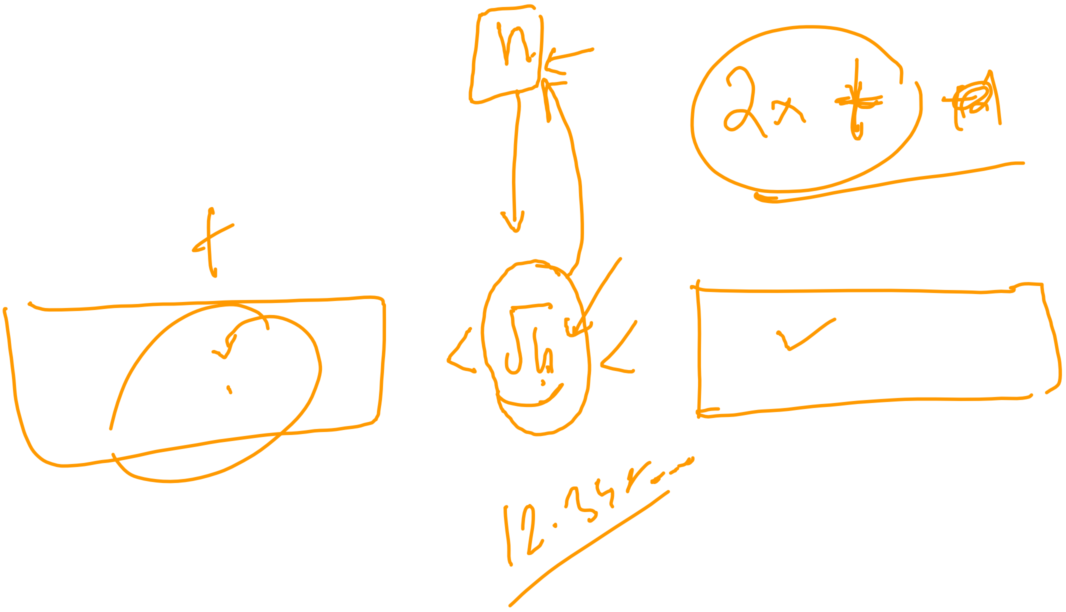
1, 2, 3, 6

9

1 3 9

24 8

$$\underbrace{1\ \ 2\ \ 3\ \ 6}\ \underbrace{4\ \ 8\ \ 12\ \ 24}$$

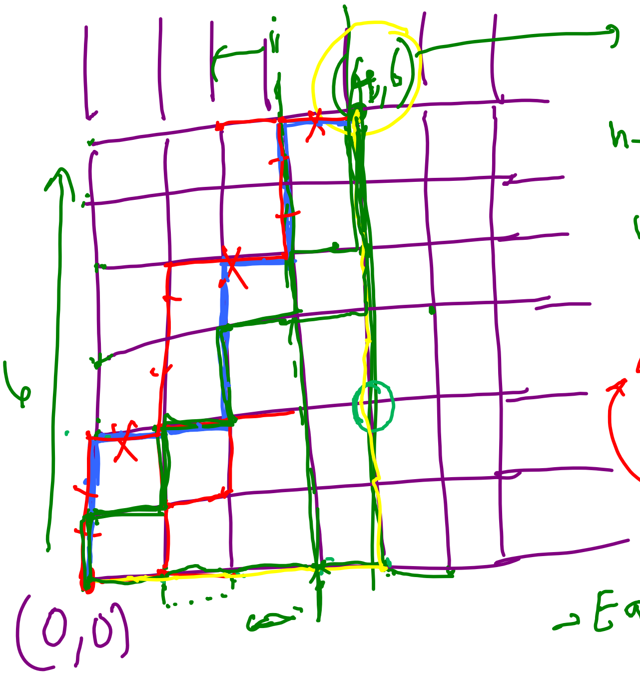


10

1 2 3 4 5 6 7 8 9 10

N N E E N E N E N E

NNEENNENNE



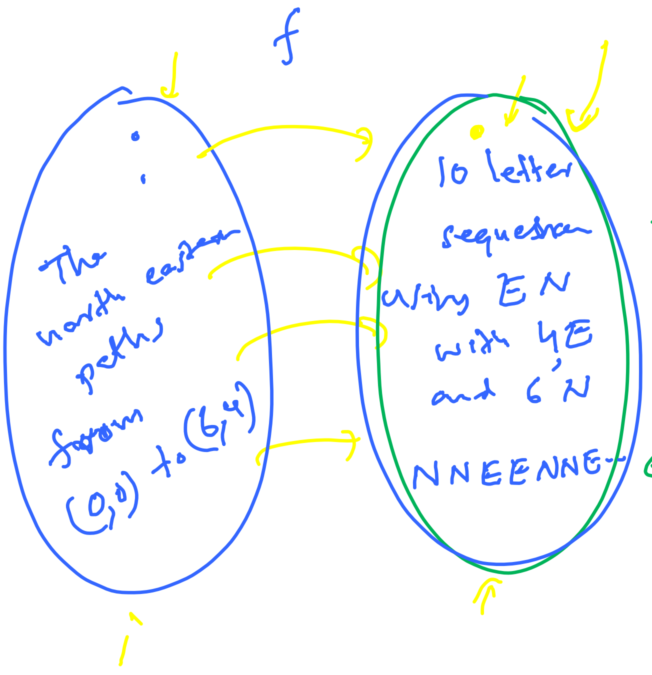
$k=4$

$n-k=6$

$n=10$

$4+6$

"10"



10

≠ E ≠ E E

10 4

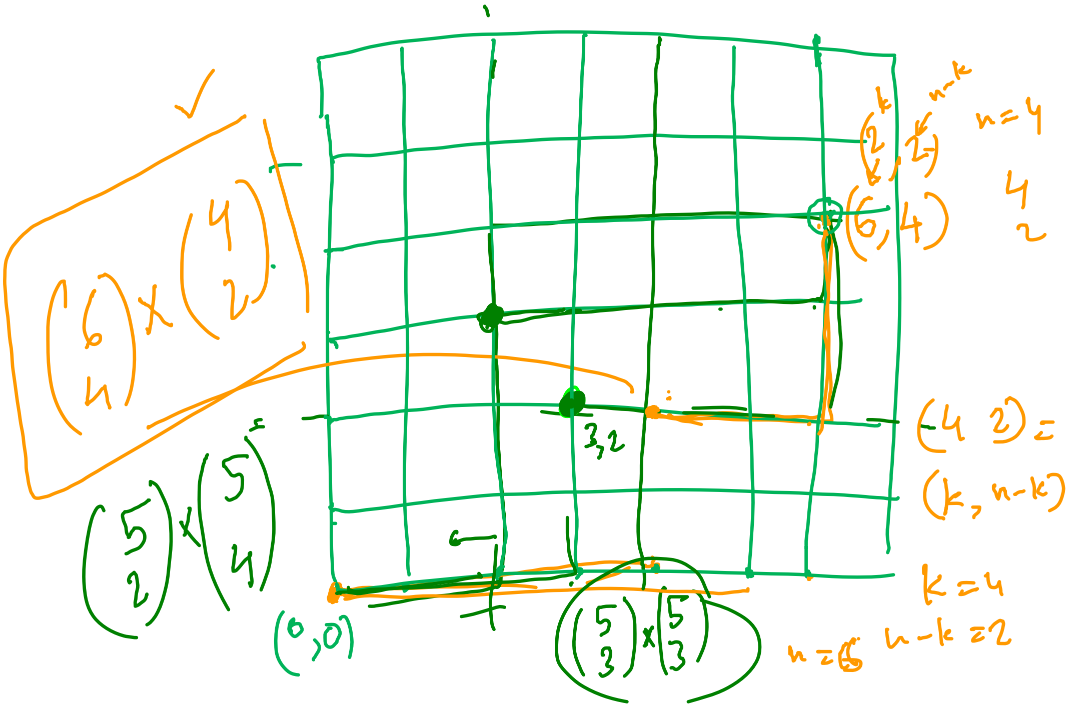
$$\binom{10}{4} \neq \binom{10}{6}$$

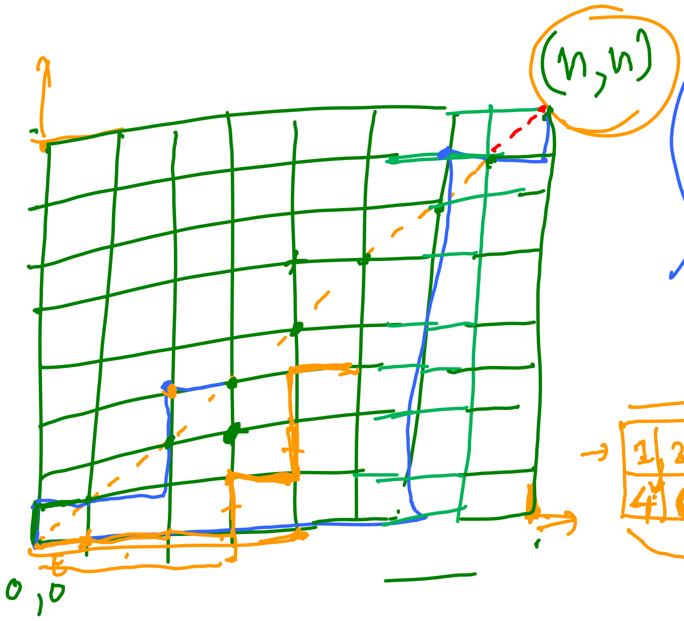
$$\underbrace{\binom{n}{k} = \binom{n}{n-k}}$$

n

$\underbrace{k}, \underbrace{n-k}$

$(0,0)$ to $(\underset{4}{k}, \underset{6}{n-k})$ $\binom{n}{k}$

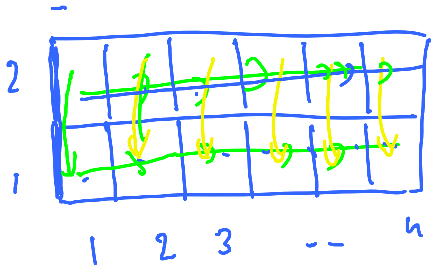




~~$\binom{2n}{n}$~~

2n

1	2	3	5	8	11
4	6	7	3	.	9

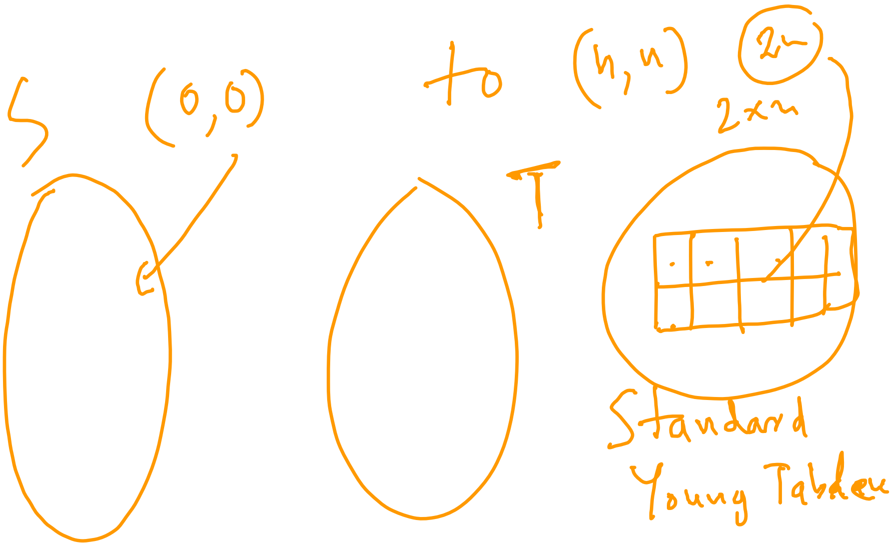


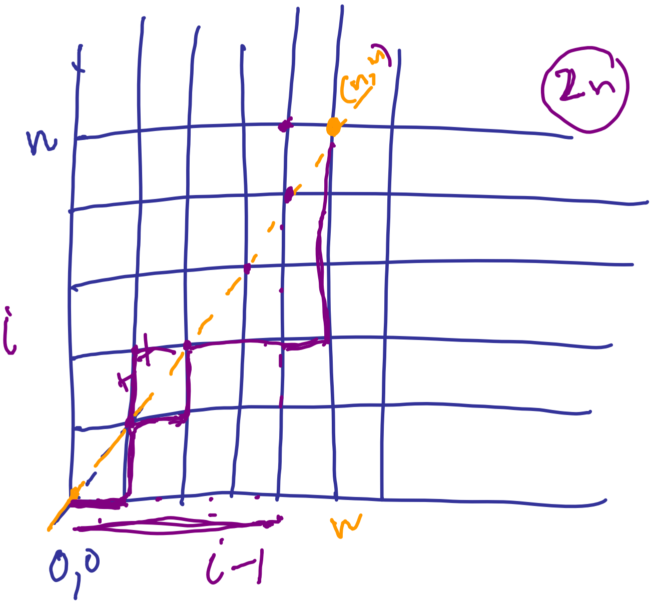
2n

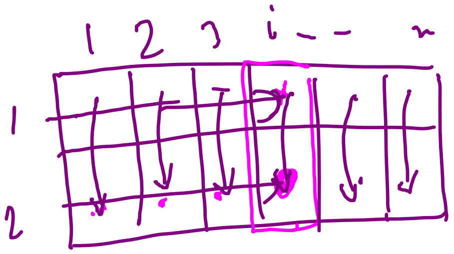
[2n]

{1, 2, ... 2n}

2	5	6	7	8
2	5	~~7~~	4	9


S
(0,0)
to
(n,n)
2n
2×n
T
Standard
Young Tableu

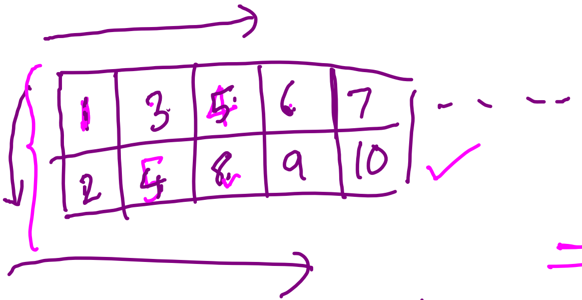

n
i
(n,n)
2n
0,0
i-1
w

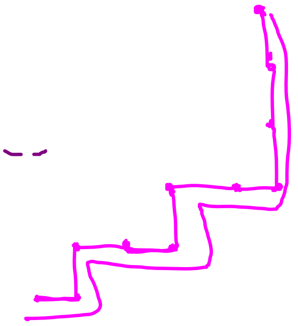


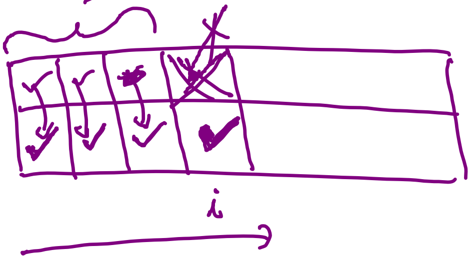
2n

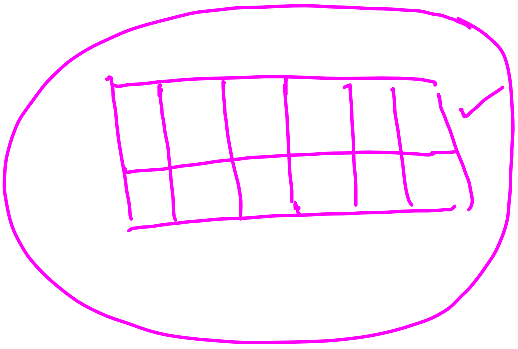
[2n]

{1, 2, 3, ----, 2n}





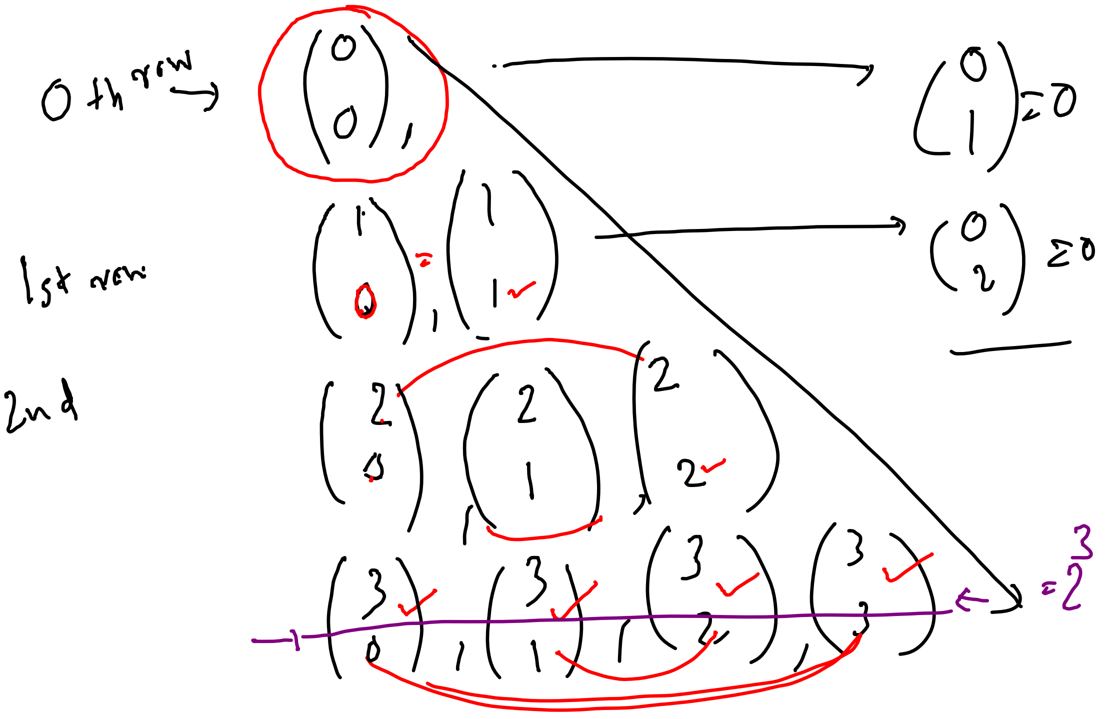




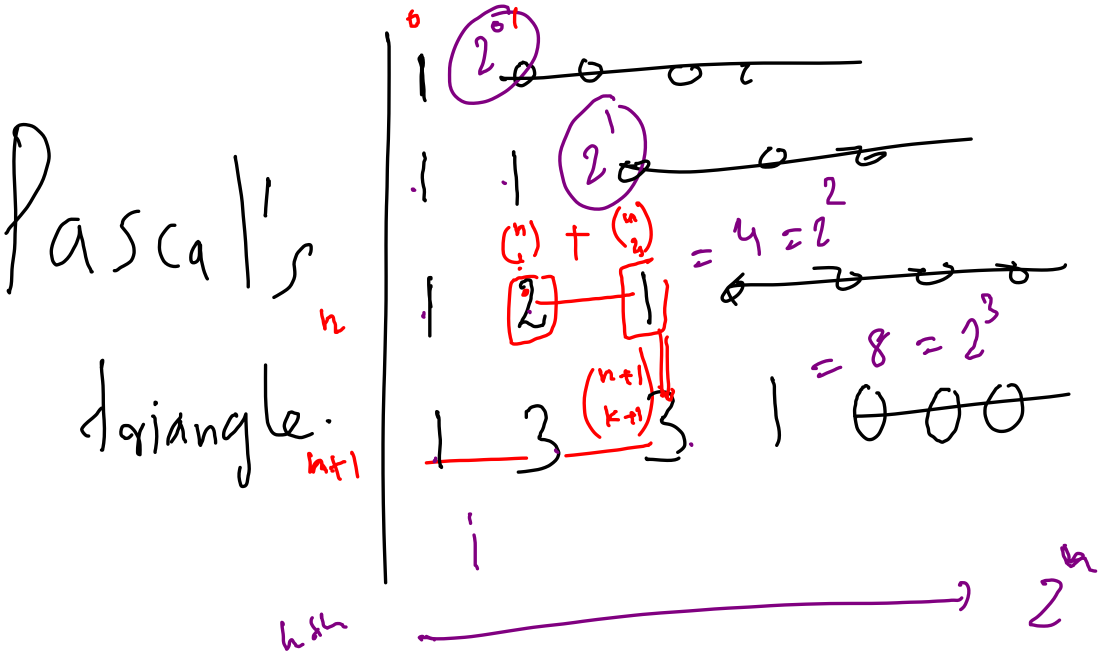
Symmetry identity $\binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}$

$$\binom{n}{n-k} = \binom{n}{k}$$

nth row $\rightarrow$ $\binom{n}{0}, \binom{n}{1}, \binom{n}{2} \dots \binom{n}{n/2} \dots \binom{n}{n-2}, \binom{n}{n-1}, \binom{n}{n}$

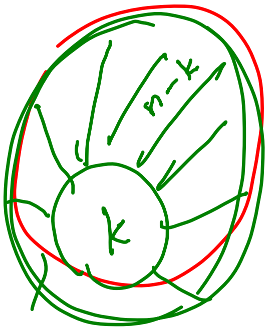


0 th row
1st row
2nd
(0 0)
(1 0) = (1 1)
(2 0) (2 1) (2 2)
(3 0) (3 1) (3 2) (3 3)
(0 1) = 0
(0 2) = 0
= 2^3

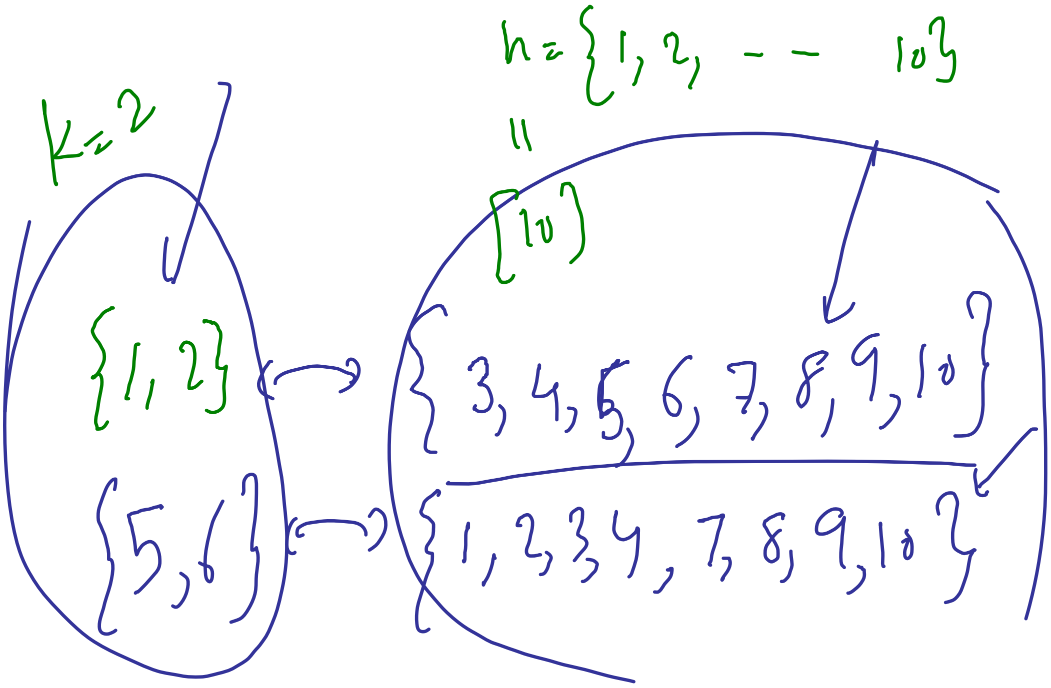


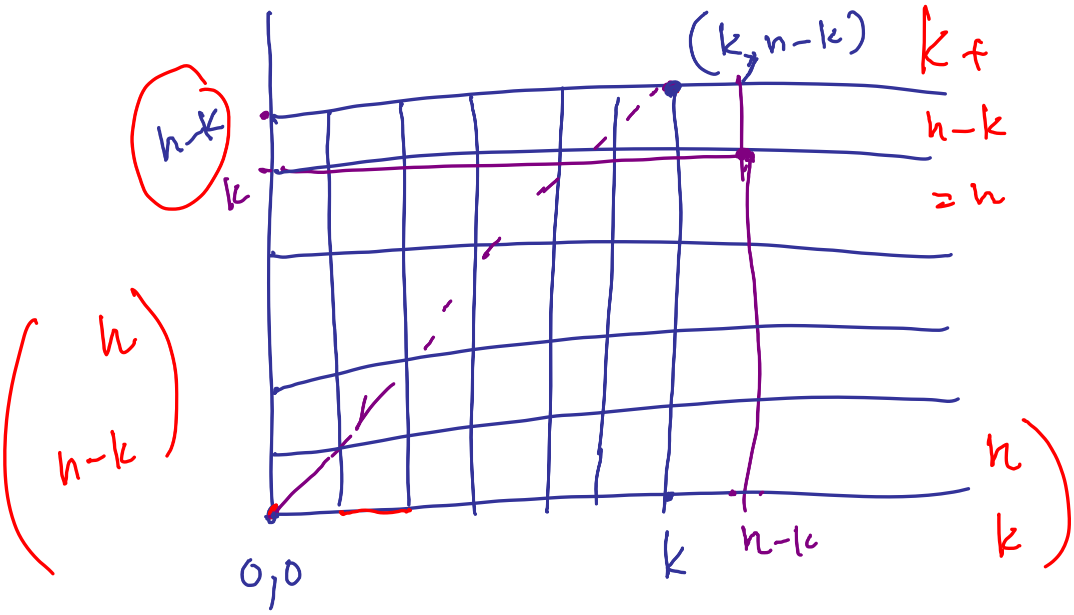
Pascal's triangle.
0
n
n+1
2^0
2^1
(n k) + (n k+1)
(n+1 k+1)
= 4 = 2^2
= 8 = 2^3
1 0 0 0 0
1 1 0 0 0
1 2 1 0 0 0
1 3 3 1 0 0 0
nth
2^n

$$\binom{n}{k} = \binom{n}{n-k}$$



# of ways to select k-element subsets from n-element universe





$(k, n-k)$ $\binom{n}{k}$

$(n-k, k)$

$\binom{n}{n-k}$

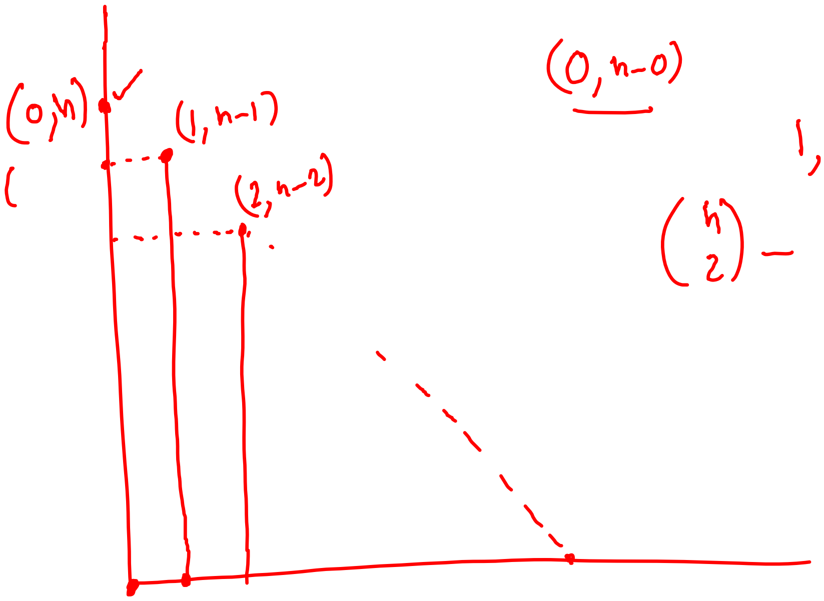
$$\underline{\binom{n}{0}+\binom{n}{1}+ - - - +\binom{n}{n}=2^n}$$

$$\xcancel{\binom{n}{n+1}=0}$$

$$\left[\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{n}\right]$$

subsets of $[n]$

$$\left\{\begin{array}{l} 1 \\ 2^n \leftarrow (1+\underset{\overset{\uparrow}{1}}{x})^n = \binom{n}{0}+\binom{n}{1}\cancel{x}+\binom{n}{2}\cancel{x^2}+\cdots \\ \qquad \cdots\binom{n}{n}\cancel{x^n} \end{array}\right\} \boxed{2^n}$$



$(0, \underline{n-0})$

1, n-1

$\binom{n}{2}$ —

$$\underbrace{\binom{n}{k} + \binom{n}{k+1}}_{} = \underbrace{\binom{n+1}{k+1}}_{}$$

addition formula

	k	(k+1)
n th arow - - - - -	$\binom{n}{k}$ +	$\binom{n}{k+1}$
n+1 th row		$\binom{n+1}{k+1}$

$\binom{0}{0}$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \rightarrow \binom{1}{0} + \binom{1}{1}$$

$$\binom{n}{0} =$$

$$\binom{n}{n} = 1$$

$$\binom{n}{k+1} + \binom{n}{k} \rightarrow \binom{n+1}{k+1}$$

1 =

1 + 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

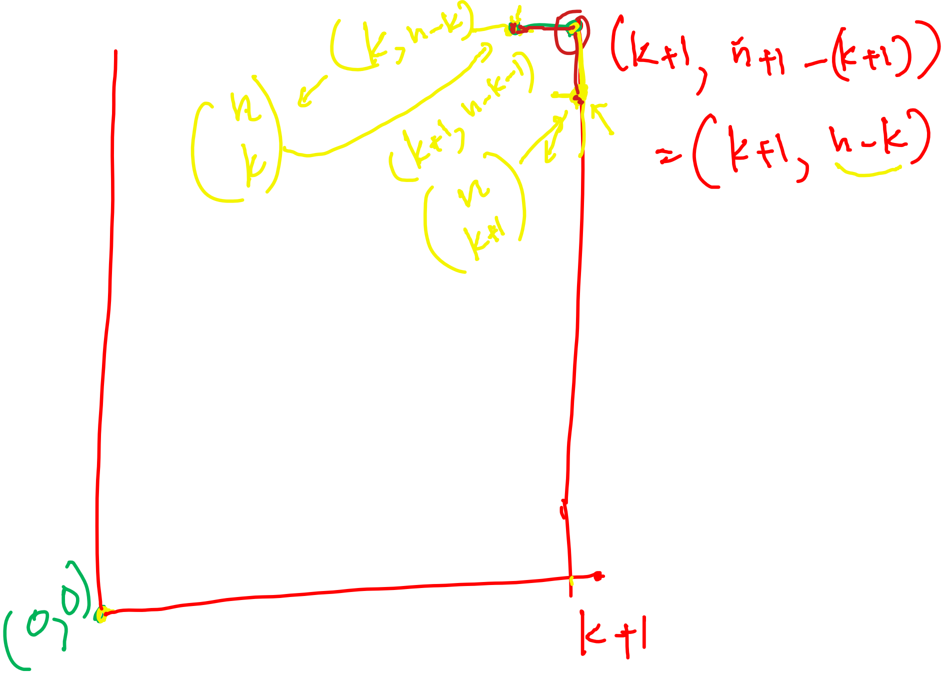
$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

n+1

$\binom{n}{k}$ [n] ↓ + $\binom{n}{k+1}$

[n+1]

[k+1]



$$\left[\sum_{k=0}^{n}\binom{n}{k}^{2}\right] = \underline{\binom{2n}{n}}$$

$n = 4$

$$\binom{8}{4}$$

$$\boxed{\underline{70}} = \frac{\overset{2}{\cancel{8}}\times 7\times \cancel{6}\times 5}{\cancel{2\cdot 3\cdot 4}}$$

$\binom{4}{0} = 1 \quad \binom{4}{1} = 4 \quad \binom{4}{2} = 6$

$\binom{4}{3} = 4 \quad \binom{4}{4} = 1$

$$1^2 + 4^2 + 6^2 + 4^2 + 1^2$$

$$1 + 16 + 36 + 16 + 1 = \underline{\underline{70}}$$

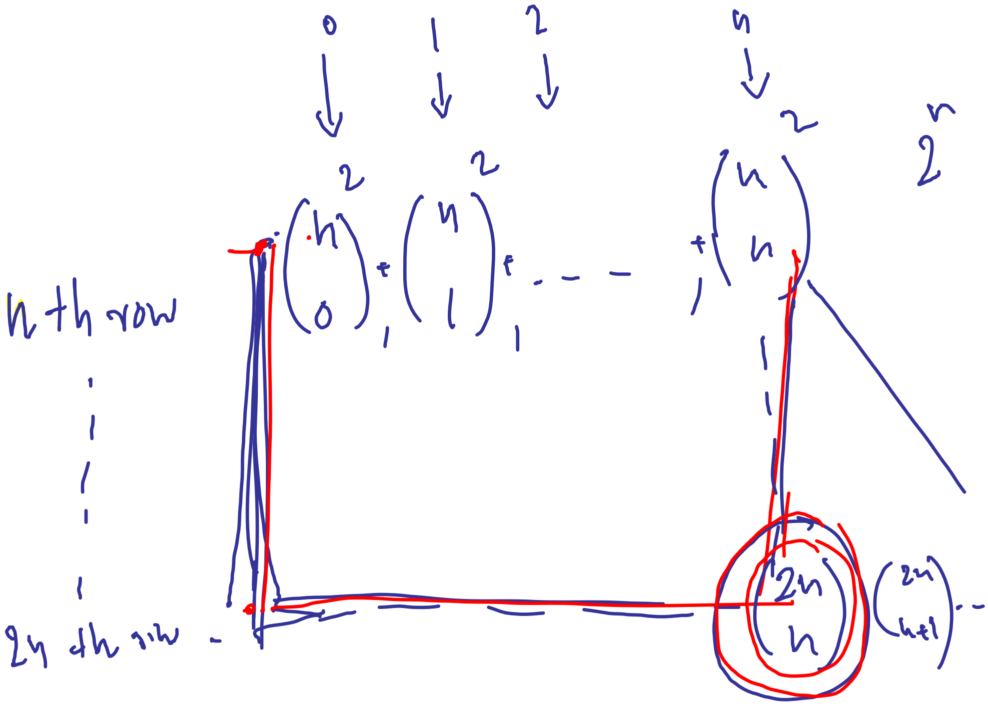
$\binom{2n}{n}$ ✓ $\qquad [2n]$ ✓

↓

"n"

$\underbrace{1, 2, \dots n}_{\text{men}} \;\Big|\; \underbrace{n+1, n+2, \dots 2n}_{\text{women}}$

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}$$



$$\binom{2n}{n} \longrightarrow [2n]$$

$$\downarrow$$

"n"

$$[2n] = \{ \underbrace{1, 2, 3, \dots n,}_{\text{men}} \; \underbrace{n+1, n+2, \dots 2n}_{\text{women}}$$

$$\left\{\begin{array}{l}\binom{n}{0}\binom{n}{\overset{0}{n}}+\binom{n}{1}\cdot\binom{n}{\overset{1}{\cancel{n-1}}}+\binom{n}{2}\binom{n}{\overset{2}{\cancel{n-2}}}\\ +\binom{n}{3}\binom{n}{\overset{3}{\cancel{n-3}}}+\cdots\binom{n}{k}\cdot\binom{n}{\overset{k}{\cancel{n-k}}}+\\ \quad \text{"}n\text{"}\quad \text{"}1\text{"}\quad n-1\quad \cdots\cdots+\binom{n}{n}\binom{n}{\overset{n}{\cancel{0}}}\end{array}\right.$$

$$\binom{n}{k}=\binom{n}{n-k}$$

$$\binom{n}{0}=\binom{n}{n}\qquad\binom{n}{1}=\binom{n}{n-1}$$

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

~~1 2 3, n+1, n+2~~ 2n-2

n = 10

$\underbrace{1, 2, 3, \ldots 9, 8}$

$$n \binom{2n-1}{n-1} = \boxed{\sum_{k=1}^{n} k \underbrace{\binom{n}{k}^2}}$$

2n persons

females | males

$\underbrace{1, 2, 3, \ldots n}$, | n+1, n+2, -- 2n

$$\boxed{n \times \binom{2n-1}{n-1}}$$

$$1\cdot\binom{n}{\textcircled{1}}\cdot\binom{n}{\underline{\overset{1}{n-1}}}+2\cdot\binom{n}{2}\cdot\binom{n}{\overset{2}{\cancel{n-2}}}$$

$$\binom{n}{n-1}=\binom{n}{1}+3\cdot\binom{n}{\underline{3}}\cdot\binom{n}{\overset{3}{\cancel{n-3}}}+\dots+k\binom{n}{\underline{k}}\binom{n}{\overset{k}{\cancel{n-k}}}$$

$$+\dots+n\binom{n}{n}\binom{n}{\overset{n}{0}}$$

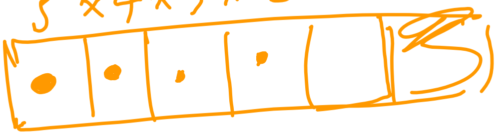
$$\left\{\begin{aligned}&1\cdot\binom{n}{1}^2+2\cdot\binom{n}{2}^2+\\&\dots k\binom{n}{k}^2+\dots+n\binom{n}{n}^2\\&=\boxed{n\binom{2n-1}{n-1}}\end{aligned}\right.$$

$5!$

$1 \times 2 \times 3 \times 4 \times 5$

$$\left\{\begin{array}{lllll} a_1 & a_2 & a_1 & a_1 & a_3 \\ \hline a_2 & a_1 & a_3 & a_5 & a_4 \\ a_4 & a_5 & a_1 & a_2 & a_3 \end{array}\right.$$

$5 \times 4 \times 3 \times 2 \times 1$



multi-sets

$$\{a_1, a_2, \ldots, a_n\}$$

$$\{\underbrace{a_1, a_1, a_1,}\ \underbrace{a_2, a_2, a_2, a_2,}\ a \ldots\}$$

$$\{\underbrace{3 \cdot a_1, 4a_2, 2 \cdot a_3}\}$$

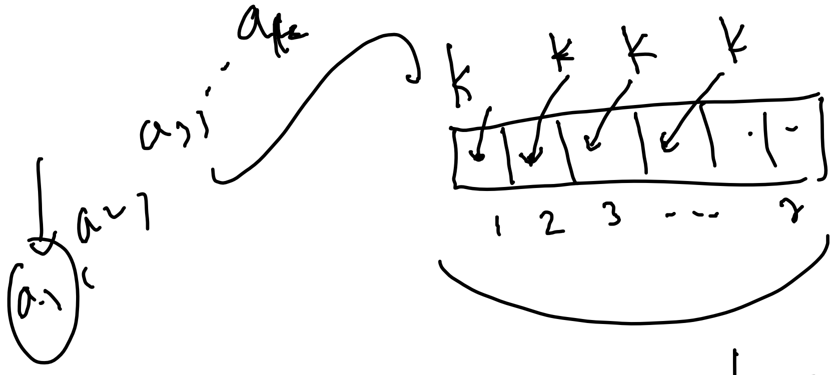
$$\frac{\left\{ \underbrace{1 \cdot a_1}_{} \ \boxed{2\, a_2} , \dots \boxed{k \cdot a_k} \right\}}{\boxed{\text{"}S\text{"}_{T}}}$$

$\curvearrowright$ r_length

r thing

$$S = \left\{ \underbrace{\infty \cdot a_1}_{} , \underbrace{\infty \cdot a_2}_{} , \dots \underbrace{\infty \cdot a_k}_{} \right\}$$

k-types of thing



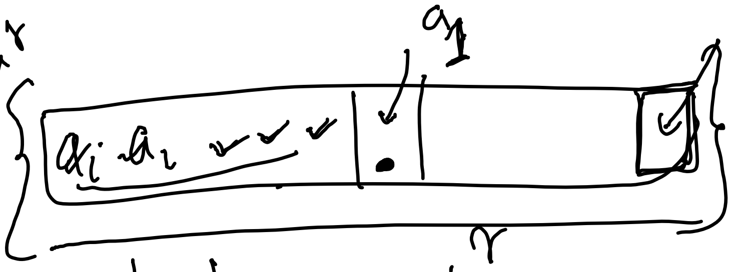
k^r

$$k \cdot k \cdot k \cdots k = k^r$$

$$S = \{\underbrace{r}\cdot a_1, \underbrace{r}\cdot a_2, \ldots, \underbrace{r}\cdot a_k\}$$

"r"

if each type has repetition at least "r"



$r = |S|$

k^r

$$S = \{ k_1 \cdot a_1, \; k_2 \cdot a_2 \; \ldots\ldots \; k_t \cdot \underbrace{a_t}_{} \}$$

$$\underbrace{|S|}_{} = \underbrace{k_1 + k_2 + \cdots + k_t}_{r = \sum_{i=1}^{t} k_i}$$

$T_1, \; T_2, \; \cdots \; T_k$

$\downarrow$

$$\left.\begin{array}{l} \#\text{ of thing of type } T_1 = n_1 \\ \#\text{ ''} \qquad\qquad T_2 = n_2 \\ \quad \vdots \\ \#\text{ ''} \qquad \text{type } T_k = n_k \end{array}\right\}$$

$$\left(\frac{n!}{n_1!\, n_2!\, n_3! \cdots n_k!} \right)$$

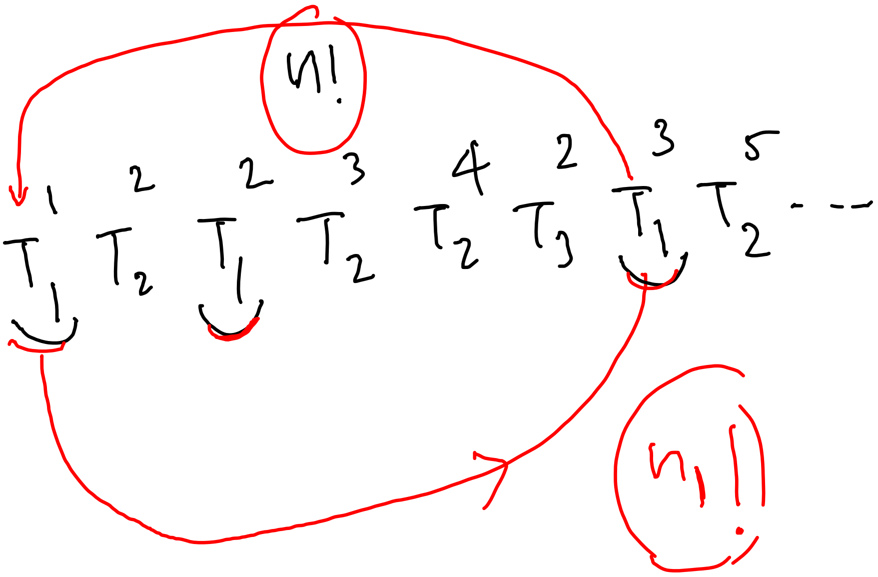
$$n_1 + n_2 + \cdots + n_k = \underset{\smile}{n} = |S|$$

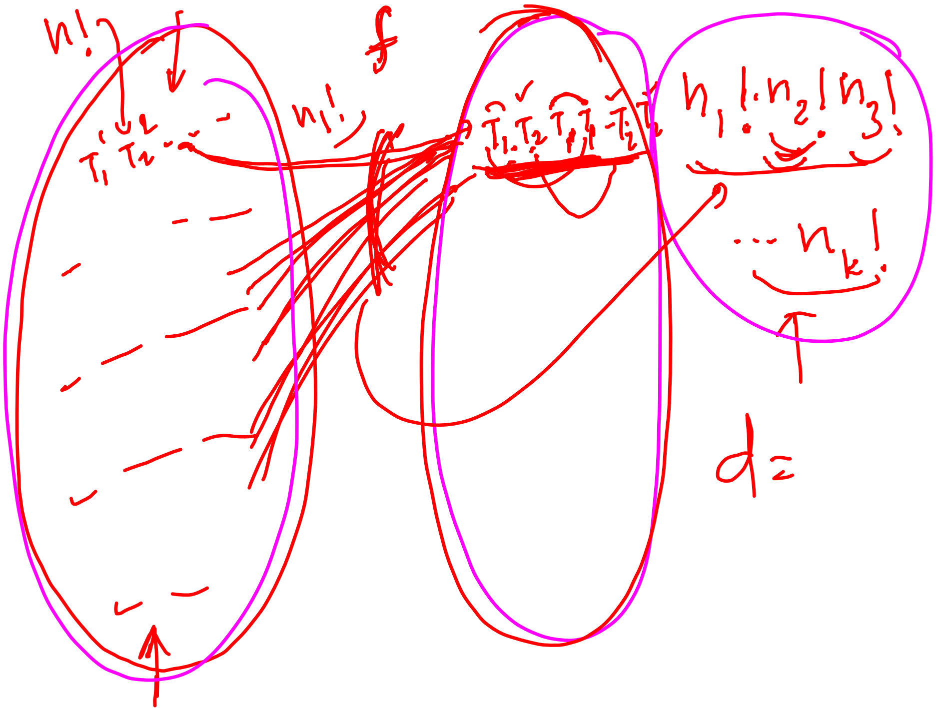
$$T_1 \to T_{\underline{1}}^{1}\ T_1^{2}\ T_1^{3}\ T_1^{4} \cdots T_1^{n_1}$$

$$T_2 \to T_2^{1}\ T_2^{2} \cdots T_2^{n_2}$$

$$\vdots$$

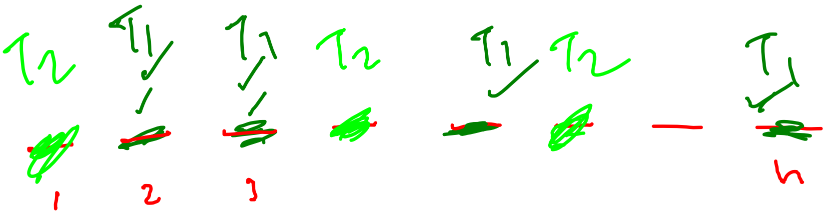
$$T_k \to T_k^{1}\ T_k^{2} \cdots T_k^{n_k}$$


n!
T_1^1 T_2^2 T_1^2 T_2^3 T_2^4 T_3^2 T_1^3 T_2^5 ...
n_1!


n!
T_1 T_2 ...
n_1!
f
T_1 T_2 T_1 T_1 T_2 T_2
n_1!·n_2!·n_3!
... n_k!
d=

$$\left(\frac{n!}{n_1!\,n_2!\,n_3!\cdots n_k!}\right)$$

d



$$S = \{n_1 \cdot T_1,\ n_2 \cdot T_2,\ n_3 T_3,\ \ldots,\ n_k T_k\}$$

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}$$

$$\binom{n-n_1-n_2-n_3}{n_4} \quad \text{"}n_2\text{"} \leftarrow \text{- - -}$$

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{\underbrace{n-n_1-n_2-\cdots n_{k-1}}}{n_k}$$

n_k

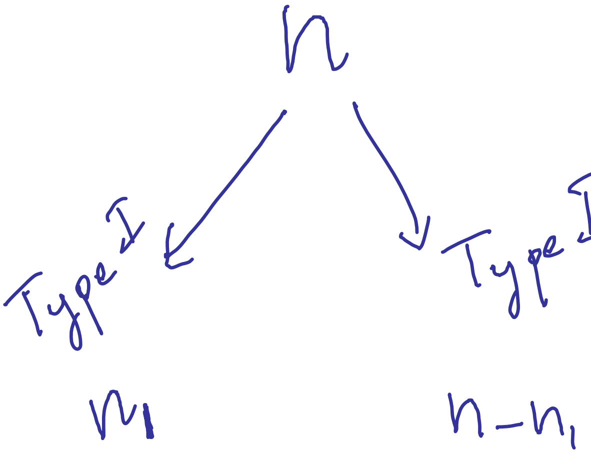
$$\frac{n!}{n_1!\,\cancel{(n-n_1)!}}\;\frac{\cancel{(n-n_1)!}}{n_2!\,\cancel{(n-n_1-n_2)!}}\;\frac{\cancel{(n-n_1-n_2)!}}{n_3!\,(n-n_1-n_2-n_3)}$$

$$|S| = n$$

$$\boxed{n_1 + n_2 + \cdots + n_k = n}$$

$$\frac{n!}{n_1!\, n_2!\, n_3! \cdots n_k!}$$

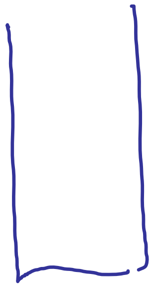
$k = 2$



$$\longrightarrow \frac{n!}{n_1!\,(n-n_1)!} = \binom{n}{n_1}$$

$$(x+y)^n \longrightarrow \boxed{\binom{n}{n_1}} x^{n_1} y^{n-n_1}$$

$$\binom{n}{\underbrace{n_1},\ \underbrace{n-n_1}} \longleftarrow$$



$$\binom{n}{n_1, n_2, \dots n_k}$$ → multinomial coefficient

$$\binom{n}{n_1} = \binom{n}{n_1, n-n_1}$$

$$(x_1 + x_2 + x_3 + \cdots + x_k)^n \; \cancel{y} \; x^0 y^0$$

$$a_1 \quad a_2 \quad a_3 \quad \quad a_k$$

$$\boxed{X} \; \underbrace{x_1^{0} \; x_2^{0} \; x_3 \cdots x_k^{0}}$$

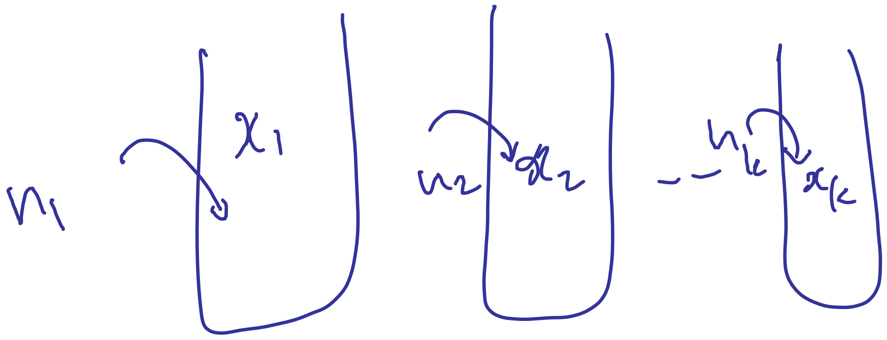
$$\overbrace{(x_1+x_2+\cdots+x_k)}^{a_1}\overbrace{(x_1+x_2+\cdots+x_k)}^{a_2}$$

$$\overbrace{(x_1+x_2+x_3+\cdots x_k)}^{a_3}$$

$$\cdots\cdots$$

$$\frac{n!}{n_1!\cdots n_k!}\,x_1^{n_1}\,x_2^{n_2}\,x_3^{n_3}\cdots x_k^{n_k} \qquad \boxed{\sum n_i = n}$$

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\cdots$$



$$\boxed{\binom{n}{n_1, n_2, \dots, n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}}$$

⑯

12

$a^{2} b^{3} c^{2} d^{5}$

$$\binom{7}{2, 2, 3} \underbrace{x^{2} y^{2} z^{2}}_{(x+y+z)^{16}}$$

$$\left(\underbrace{a}_{w} + \underbrace{2b}_{x} + \underbrace{\boxed{3}c}_{y} + \underbrace{\boxed{2}d}_{z} + \underbrace{5}_{v}\right)^{16}$$

$$(w+x+y+z+v)^{16}$$

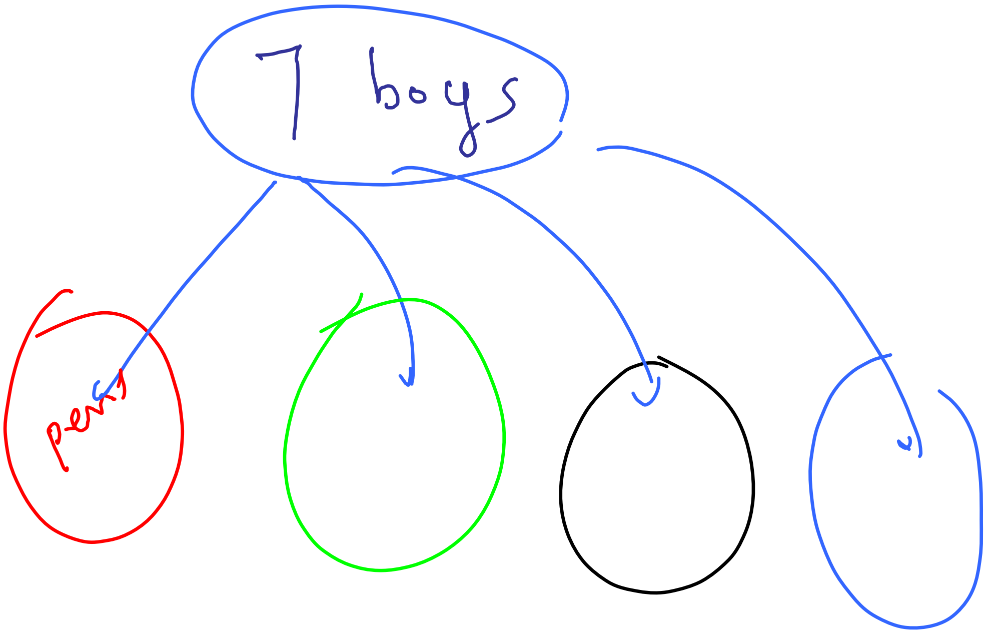
$$\binom{16}{2,3,2,5,4} = \frac{16!}{2!,3!,2!,5!,4!} \xleftarrow{} w^2 x^3 y^2 z^5 v^4$$

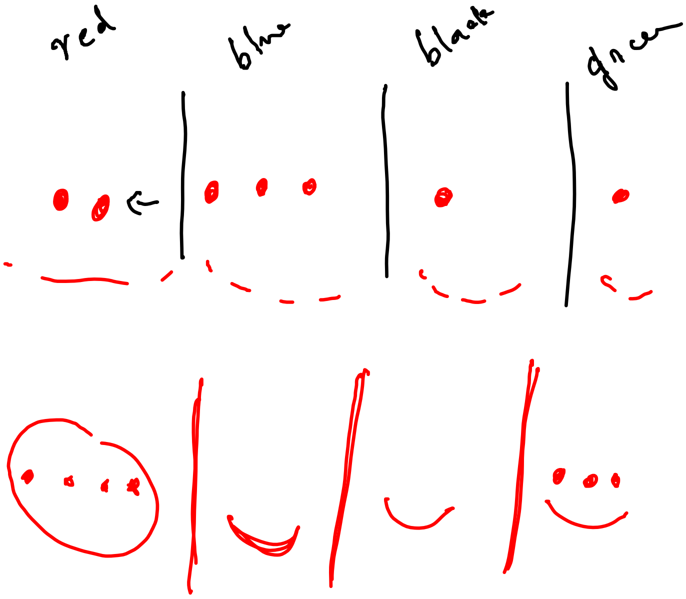
12 ④

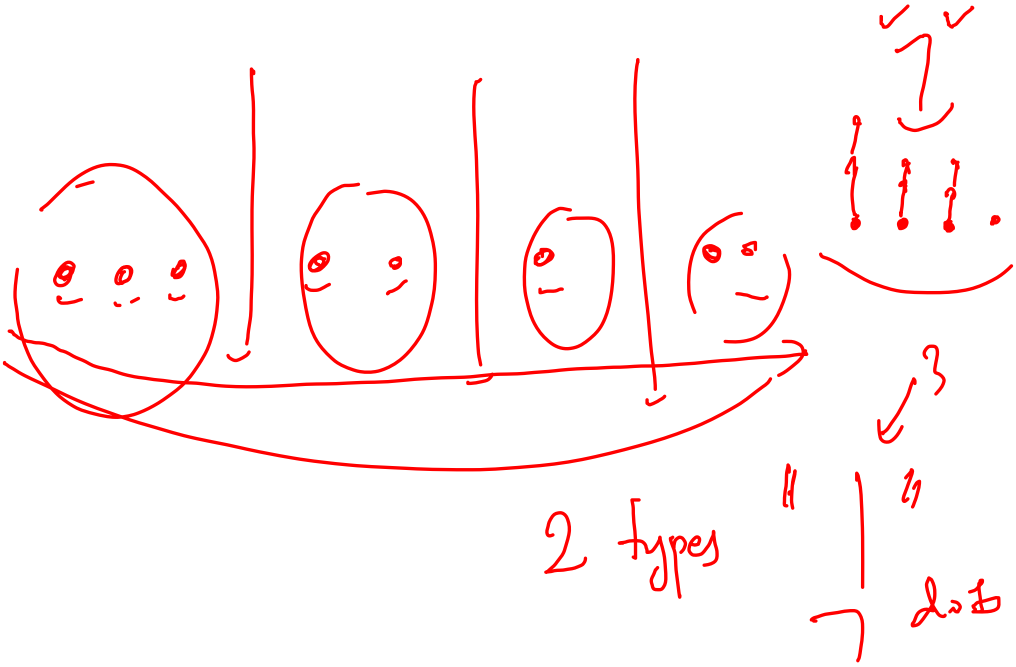
$$\underbrace{n_1+n_2+n_3+n_4+n_5}_{} = 16$$

$$\overline{(2h)^3}$$

$$\left(\frac{16!}{2!,3!,2!,5!,4!}\right) \times 8 \times (-3)^2 \times 2^5 \times 5^4$$


7 boys
pens


red
blue
black
green



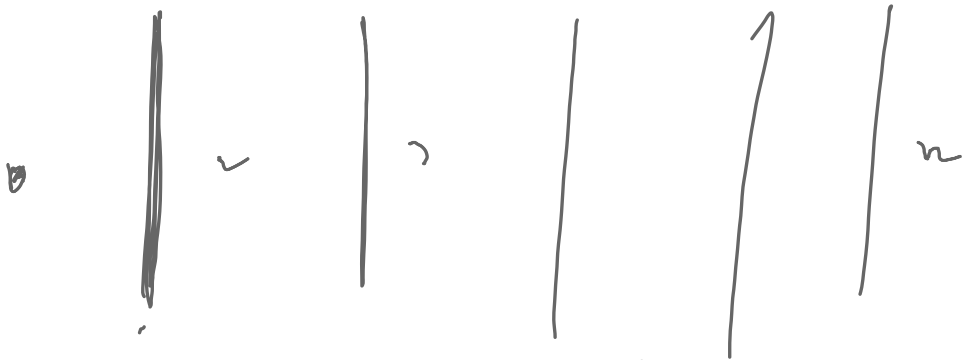
$$\frac{(7+3)!}{7!\,3!} = \binom{10}{7,3}$$

r boys "r"

$\underline{n \text{ types}}$

$\underbrace{T_1, T_2, T_3, T_i \text{ --- } , T_n}$

r



r dots & $n-1$ bars

$$\frac{(n+r-1)!}{r!\,(n-1)!} = \frac{\binom{n+r-1}{r}}{\binom{n+r-1}{n-1}}$$

$$\binom{r+n-1}{r}$$

$r = \underbrace{12}$ -w

$n = 20$

$$= \binom{12+20-1}{12} \nearrow = \binom{31}{12}$$

$$\frac{n=4}{r=10} \quad \binom{10+4-1}{10}$$

$$= \binom{r+n-1}{r}$$

$$10 \text{ rs} \longrightarrow \cancel{4}$$

6 rs

6

$$\binom{9}{6}$$

n = 4

4 n

$$\binom{r+n-1}{r} = \binom{6+3}{6}$$

$10 - 5 = 5 - 3$

$\binom{2+4-1}{2} = \text{"}2\text{"}$

"3"

B, C, D

$n=4$

$r=2$

$= \binom{5}{2}$

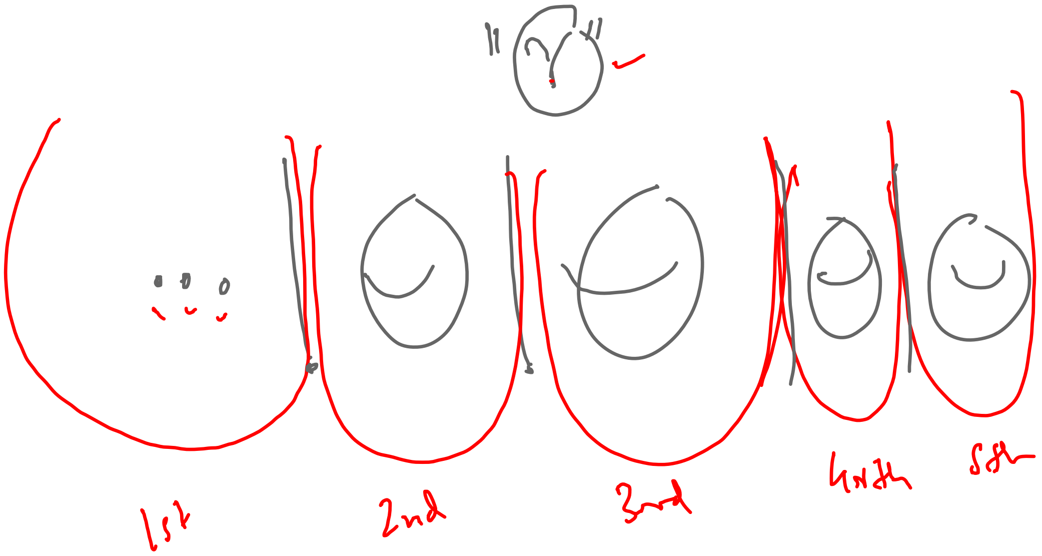
$=$ A, B, C, D

r $\quad$ $\underbrace{n}$

$\binom{r-1}{r-n}$ $\quad$ $\underbrace{r-n}$

$= \binom{r-1}{n-1}$ ✓ $\binom{(r-\cancel{n})+\cancel{n}-1}{r-n}$

$$\binom{r-1}{n-1}$$



$$\binom{r+n-1}{r}$$

$$\binom{r-1}{n-1}$$

$$\begin{bmatrix} 7 \text{ bananas} \\ 6 \text{ oranges} \end{bmatrix}$$

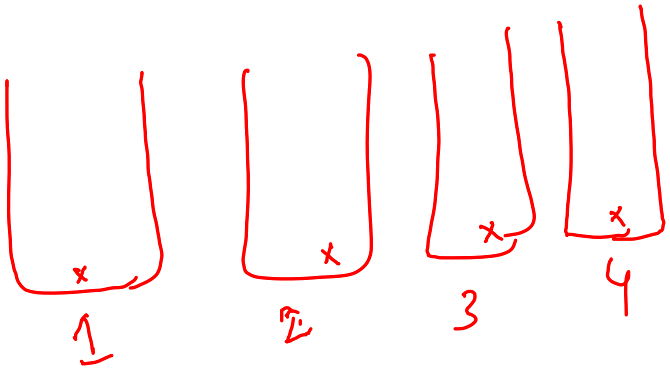
$$4 \text{ children} \longrightarrow \left\{ \begin{array}{l} \text{each child} \\ \text{should get} \\ \text{at least 1 banana} \end{array} \right.$$

$\binom{3+4-1}{3}$

$\underbrace{7 \text{ bananas}}_{4 \text{ children}}$

7 − 4 = 3

$\binom{6}{3}$



$\binom{\overset{7}{r}-1}{\overset{4}{n}-1}$

$r = 7$

$n = 4$ boxes

$\binom{6}{3}$

6 oranges } identical balls

4 children } 4 boxes

$$\binom{6+4-1}{6} = \binom{9}{6}$$

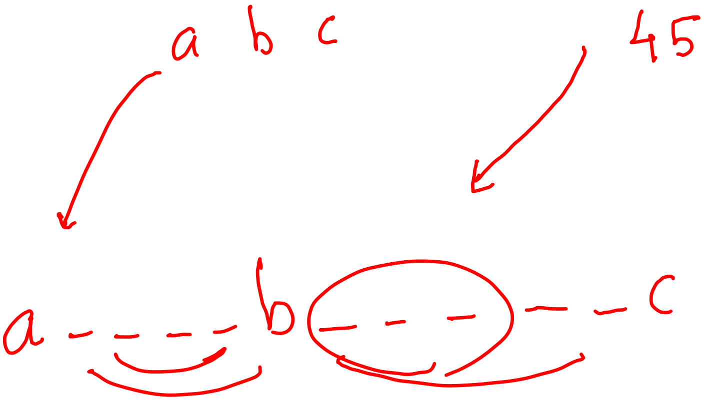
$$\underline{\binom{6}{3} \times \binom{9}{6} =}$$

12

a b c

a b c
a c b
b c a
b a c
c a b
c b a

6 ways



$a_1 \quad a_2 \quad a_3 \quad \cdots \quad a_{12}$

$12! \longrightarrow$ For each message,

$a_1 \overset{\cdots}{\_} a_2 \overset{\cdots}{\_} a_3 \overset{\cdots}{\_} a_4 \cdots \_ a_{11} \overset{\cdots}{\_} a_{12}$

11 gaps

$3 \times 11 = 33$ spaces

$45 - 33 = \boxed{12} \checkmark$

$\overset{1}{\downarrow} \qquad \overset{2}{\downarrow} \qquad \overset{3}{\downarrow} \qquad \overset{11}{\downarrow}$

$a_1 \cdots \checkmark \quad a_2 \checkmark \quad a_3 \checkmark \quad a_4 \cdots \checkmark \; a_{11} \checkmark \; a_{12}$

$n = 11 \qquad r = 12$

$$\binom{12+11-1}{12} = \binom{22}{12}$$

$$(12)!\binom{22}{12} = 12!\binom{22}{12}$$

① n <u>objects</u>, r

$$\binom{r+n-1}{r}$$

(• • • • •)

(2) $\textcircled{r}$ ↙ identical balls

$\textcircled{n}$ labelled boxes

$$\binom{r+n-1}{r}$$

$$\binom{r-1}{n-1} \checkmark$$

(3)

$$\boxed{x_1 + x_2 + \cdots + x_n = r}$$

$$x_i \geq 0$$

$$x_1 + x_2 + x_3 + x_4 = 7$$

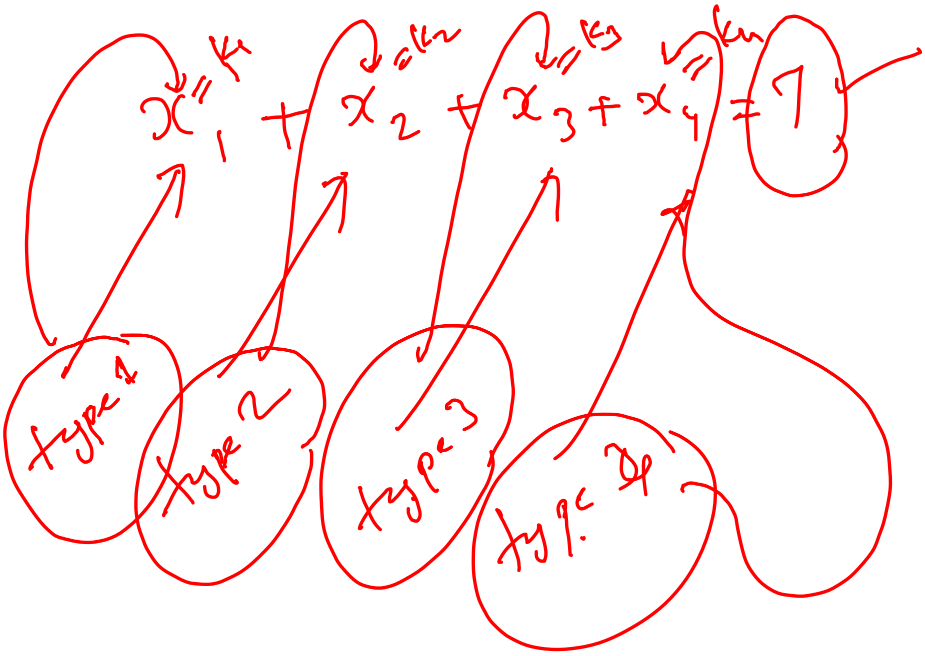
$$0 + 1 + 2 + 4 = 7 \checkmark$$

$$1 + 0 + 2 + 4 =$$

$$x_1 + x_2 + x_3 + x_4 = \boxed{7} = r$$

$$\underbrace{\binom{r+n-1}{r}}_{}=\binom{7+4-1}{7}$$

$$=\binom{10}{7}$$



$$x_1 + \cdots + x_n = r$$

$$x_i \geq 0$$

$$\boxed{\binom{r+n-1}{r}} \leftarrow$$

$$x_1 + x_2 + \cdots + x_6 < 10$$

$$\underline{x_1 + \cdots + x_{\underset{n}{6}} = \overset{r}{0}} \checkmark$$

$$\underline{x_1 + \cdots + x_6 = 1} \checkmark$$

$$\underline{x_i \geq 0}$$

$$\binom{1+6-1}{0} = \binom{5}{0} = 1$$

$x_1 + x_2 + \cdots + x_6 = 2$ $\quad \binom{0 + 6-1}{0} = \binom{5}{0}$

$\binom{1 + 6-1}{1} = \binom{6}{1}$

$x_1 + \cdots + x_6 = 3$

$\vdots$

$x_1 + \cdots + x_6 = 9$ $\quad \binom{3 + 6-1}{2} = \binom{\cancel{7}^{8}}{2}$

$\underbrace{x_1 + x_2 + \cdots + x_6}_{\cancel{10}\ \checkmark} + \boxed{x_7} = \boxed{<}\ 10$ ✓

$\underbrace{x_1 + x_2 + \cdots + x_6 + x_7}_{\checkmark} = 10$ ✓

$\boxed{x_7 \geq 1}$

$\boxed{x_7 = 0}$ ✓

$$\boxed{x_7}$$ ✓

$$x_1 + x_2 + \cdots + x_6 + x_7 = 10$$

$$x_i \geq 0 \quad 1 \leq i \leq 6$$

$$\boxed{x_7 \geq 1}$$ ✓

$$\begin{tikzcd}
x_1 \arrow[d] & + & x_2 \arrow[d] & + \cdots + & x_6 & + & x_7 \arrow[d] & = & 10 \arrow[d] & \text{"9"} \\
y_1 & + & y_2 & + \cdots + & y_6 & + & y_7 & = & 9 & = r
\end{tikzcd}$$

$$= n$$

$$y_7 = x_7 - 1$$

$$y_i \geq 0$$

$$\binom{9+7-1}{9} = \underline{\underline{\binom{15}{9}}}$$

$$(w+x+y+z)^{10}$$

11 terms

$$(x+y)^{10} = \underbrace{\overbrace{\binom{10}{0}x^0y^{10}}}_{} + \underbrace{\overbrace{\binom{10}{1}x^1y^9}}_{}$$

$$+ \cdots + \underbrace{\binom{10}{10}x^{10}y^0}_{}$$

$(x+y)^n$

$\underline{n+1}$

$\binom{n+2-1}{n}$

r

$x^i y^{n-i}$

$x^a y^b$

$a+b=n$

$x_1+x_2=n$

$x_i \geq 0$

$i+n-i=n$

$\binom{n+1}{n} = \binom{n+1}{1} = \underline{n+1}$

$(w+x+y+z)^n$ (y

$n_1 \quad n_2 \quad n_3 \quad n_4$

$w \quad x \quad y \quad z$

$\binom{n+4-1}{n}$

$n_1+n_2+n_3+n_4 = n = r$

$\boxed{n_i \geq 0}$

$$(x_1+x_2+\cdots+x_k)^{\textcircled{n}}$$

$$\binom{n+k-1}{n} \leftarrow x_1^{n_1}\, x_2^{n_2} \cdots x_k^{n_k}$$

$\underline{n_i \geq 0}$

$\underline{n_1+n_2+\cdots+n_k = \textcircled{n}}$

$$\boxed{1 = \textcircled{1}} \leftarrow 2^0 = 2^{\textcircled{1}-1}$$

$$\boxed{\begin{array}{c} 1+1 = \textcircled{2} \\ 2 = 2 \end{array}} \leftarrow 2^1 = 2^{\textcircled{2}-1}$$

$$\boxed{\begin{array}{c} 3 = \textcircled{3} \\ 1+2 = 3 \\ 2+1 = 3 \\ 1+1+1 = 3 \end{array}} \quad 2^2 = 2^{\textcircled{3}-1}$$

$4 = 4$ ✓

$1 + 3 = 4$ ✓ $\quad 2 + 2 = 4$

$3 + 1 = 4$ ✓

$2^{4-1} = 2^{3} = 8$

"8"

$1 + 1 + 2 = 4$ ✓

$1 + 2 + 1 = 4$ ✓ → $\underline{2} + 1 + 1 = 4$

$1 + 1 + 1 + 1 = 4$ ✓

$$\begin{cases} n_1 + n_2 + \dots + n_k = n \\ n_2 + n_1 + \dots = n \end{cases}$$

$\underline{n_i \geq 1}$

$$x_1+x_2+\dots+x_k=n$$

$$\binom{n+k-1}{n}$$

$$x_i \geq 0$$

$k=1$

$k=2$

$\vdots$

$k=n$

$$x_1=n \leftarrow \binom{n-1}{0} \qquad \binom{n-1}{1}$$

$$x_1+x_2=n$$

$$y_1+y_2=n-2 \qquad \binom{n-2+2-1}{n-2}$$

$$y_1=x_1-1 \qquad y_2=x_2-1 \qquad =\binom{n-1}{n-2}$$

$$\binom{n-1}{n-3} = \binom{n-1}{2}$$

$$\|$$

$$\binom{n-\cancel{3}+\cancel{3}-1}{n-3}$$

$$\boxed{x_1 + x_2 + x_3 = n}$$

$$\underline{x_i \geq 1}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$y_1 + y_2 + y_3 = \underline{n-3} \checkmark$$

$$\underline{y_i \geq 0}$$

$$\underline{x_i \geq 1}$$

$$\underline{x_1 + \cdots + x_k} = \underline{n}$$

$$\downarrow$$

$$\boxed{y_1 + \cdots + y_k = n-k}$$

$$y_i \geq 0$$

$$\boxed{\binom{n-1}{k-1}} \checkmark$$

$n-1$

$$\binom{n-1}{n-1-(n-k)} = \binom{n\cancel{-k}+\cancel{k}-1}{n-k}$$

$$\underbrace{2^{n-1}}_{} = \underbrace{\binom{n-1}{0}+\binom{n-1}{1}+\cdots+\binom{n-1}{k-1}+\underbrace{\binom{n-1}{n-1}}_{}}_{}$$

$$\overset{1}{x_1}+\overset{1}{x_2}+\cdots+\overset{1}{x_n}=\underline{n}$$

10

$\cancel{1+9=10}$ $\quad \cancel{{}_0\int d}$

$2+4+1+1=10$

$\underbrace{2+2+2+2+2}=10 \swarrow \boxed{2n}$

$6+4=10$

$10=10$

~~2 + 4 +~~ $\boxed{7}$

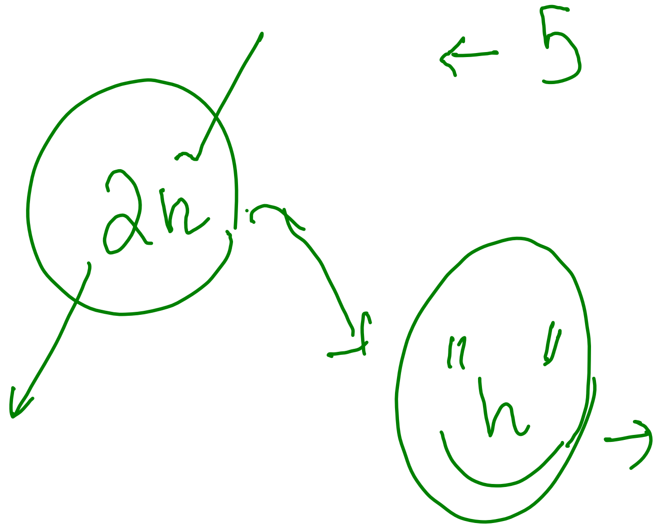
$10 = 2 \times 5$ $\boxed{2n}$

$10 = 10$

$2 \times (5) = 10$

$4 + 6 = 10 \Rightarrow \cancel{2}\left(\overset{5}{\underbrace{2+3}}\right) = 10$

$4 + 4 + 2 = 10 \Rightarrow \cancel{2}\left(\underbrace{2+2+1}\right) = 10$



$2^4 = \underline{16}$

2^{n-1}

```
i=1 to n ← 20
    j=1 to i
        print (sth)
```

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$i=1$ to n

$j=1$ to i

$k=1$ to j

$$\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}(1) = ?$$

$i = a \quad j = b \quad k = c$

$[n]$

$$\underline{a \geq b \geq c}$$

$$(a, b, c)$$

i, j, k ✓ ✓ ✓

$i \geq j \geq k$　　$1 \leq a, b, c \leq n = 20$

(a, b, c)　　$a \geq b \geq c$

$i = a$ ✓　　$b \leq a$

$j = b$ ✓

$c \leq b \leq a$

$k = c$

$\overset{i}{(a}, \overset{j}{b}, \overset{k}{c})$

"n"

$[n]$

$\binom{n}{3}$

$\Big\{ \{x, y, z\},\ x \geq y \geq z$ and $1 \leq x, y, z \leq n \Big\}$

$$\boxed{\overset{a}{x} > \overset{b}{y} > \overset{c}{z}}\ ✓$$

$r = 3$

$k = n$

$$\binom{3+n-1}{3} = \binom{n-2}{3}$$

x, y, z

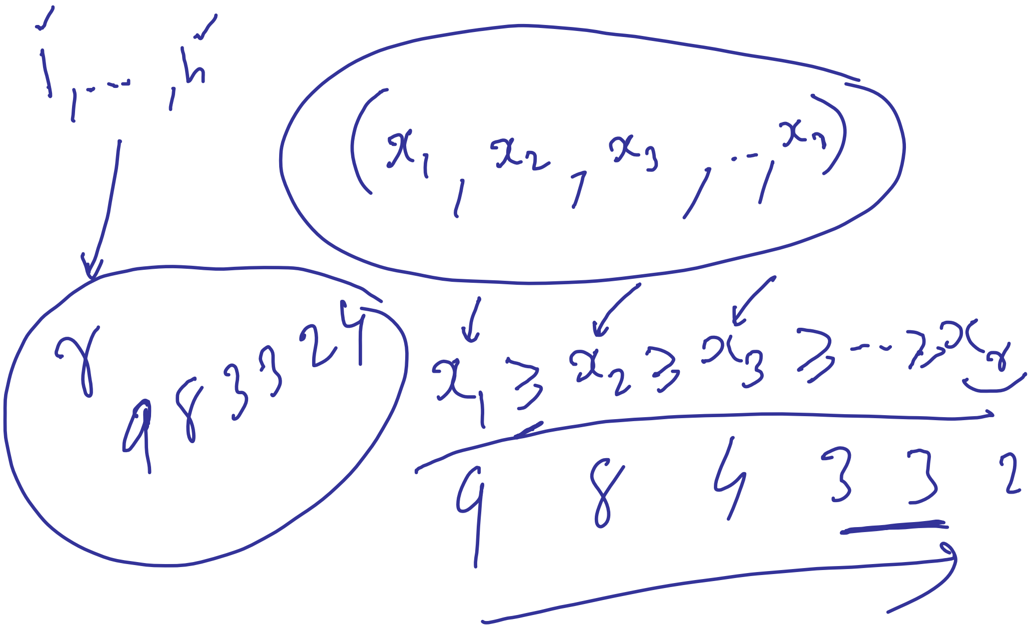
n

$1, 2, 3, \ldots n$

x	y	z
3	2	2

2 2 3

```
for $i_1 = 1$ to $n$
for $i_2 = 1$ to $i_1$
for $i_3 = 1$ to $i_2$
      ⋮
for $i_r = 1$ to $i_{r-1}$
        print ( — )
```



$$\boxed{\binom{\overset{2}{r}+n-1}{2 \nearrow r}}$$

$\underline{\underline{r=2}}$

$\boxed{1+2+3+\cdots+n}$

```
for i=1 to n
    for j=1 to i
        print;
```

$$\binom{2+n-1}{2} = \underbrace{\binom{n+1}{2}}_{\frac{(n+1)n}{2}} = \underbrace{\overset{\checkmark}{1}+\overset{\checkmark}{2}+\overset{\checkmark}{3}+\cdots+\overset{\checkmark}{n}}_{\frac{n(n+1)}{2}}$$

EEEEE OOOOOOO~~O~~ O -

7 run

5 Es

~~10 Os~~

O EEE O E O O E O O O -

E O EE OO EE OOO EE

4 "E"-runs

3 "O" runs

O...O E...E O...O EE...E

... EEE OOO

$y_i = x_i - 1$

EE OE ~~E~~ OE EE

x_1 x_2 x_3 x_4 ... x_7

$y_i \geq 0$

$\boxed{x_i \geq 1}$

$x_i \geq 0$

$$\begin{cases} \overset{y_1 + y_3 + y_4 + y_5}{x_1 + x_3 + x_4 + x_7} = 5 - 4 = 1 \\ x_2 + x_4 + x_6 = 10 \\ \underbrace{y_2 \quad y_4 \quad y_6} = \textcircled{7} \end{cases}$$

$y_i = x_i - 1$

$$\left\{ \underline{y_1 + y_2 + y_3 + y_4 = 1} \right.$$

$$y_i \geq 0$$

$$\begin{pmatrix} 1 + 4 - 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 7 + 3 - 1 \\ 7 \end{pmatrix} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} 9 \\ 7 \end{pmatrix}}$$

$$\underbrace{OO}_{x_1}\ \underbrace{EE}_{x_2}\ \underbrace{OOO}_{x_3}\ \underbrace{EE}_{x_4}\ \underbrace{OOO}_{x_5}\ EE_{x_1}$$

$x_4 \downarrow OOO$ x_7

$$\boxed{x_i \geq 1} \rightarrow y_i \geq 0$$

$$\cancel{x_1}\,y_1 + \cancel{x_3}\,y_3 + \cancel{x_5}\,y_5 + \cancel{x_7}\,y_7 = \cancel{10}\ 6$$

$$\cancel{x_2}\,y_2 + \cancel{x_4}\,y_4 + \cancel{x_6}\,y_6 = \cancel{5}\ 2$$

$$\binom{6+4-1}{6} = \boxed{\binom{9}{6} \checkmark \times \binom{4}{2} \checkmark}$$

$$\binom{2+3-1}{2} = \binom{4}{2}$$

$$\underbrace{\text{"}n\text{"}}$$

$$\underline{r} \qquad \text{"}r\text{"}$$

$$\underline{\binom{r+n-1}{r}}$$

$$2^{\boxed{10^{10}}}$$

$$\binom{100}{20} = \frac{100!}{20!\,80!}$$

$$\binom{n}{r}$$

$$\left(\frac{n}{r}\right)^r \leq \underbrace{\binom{n}{r}}_{} \leq \left(\frac{en}{r}\right)^r$$

$$\left(\frac{n}{r}\right)^r \qquad \boxed{e^r}\left(\frac{n}{r}\right)^r$$

$$\left(\frac{n}{r}\right)^r \leq \binom{n}{r}$$

$$\underbrace{\boxed{\frac{n}{r}} \cdot \frac{n}{r} \cdot \frac{n}{r} \cdots\cdots \frac{n}{r}}_{r \text{ times}} \leq \boxed{\frac{n}{r}}\,\boxed{\frac{n-1}{r-1}}\;\frac{n-2}{r-2}\;\frac{n-3}{r-3}\;\left(\frac{n-r+1}{1}\right)$$

$$\binom{n}{r}$$

$$\frac{n}{r} \le \frac{n-1}{r-1} \le \frac{n-2}{r-2} \le \frac{n-3}{r-3} \le \cdots \le \frac{n-r+1}{\cancel{r}\,1} \qquad \binom{n}{r}$$

$$n(r-1) \le r(n-1)$$

$$\cancel{nr} - n \le \cancel{rn} - r$$

$$\boxed{r \le n}$$

$$\left(\frac{n}{r}\right)^r \le \frac{n(n-1)\cdots(n-r+1)}{r!} = \binom{n}{r}$$

$$\binom{n}{r} \le \left(\frac{en}{r}\right)^r$$

$$\boxed{(1+t)^n \le e^{tn}}$$

$$e^{tn} \ge (1+t)^n = \binom{n}{0} t^0 + \binom{n}{1} t^1 + \dots + \binom{n}{k} t^k + \dots + \binom{n}{n} t^n$$

$$e^{t n \,\overset{\frac{r \times n}{n}}{}} \ge (1+t)^n \ge \underbrace{\binom{n}{r}\left(\frac{r}{n}\right)^r}$$

$$e^{r} \ge \binom{n}{r}\left(\frac{r}{n}\right)^r \qquad \boxed{t = \frac{r}{n}}$$

$$\boxed{\begin{aligned}\binom{n}{r} &\le e^{r}\left(\frac{n}{r}\right)^{r} \\ &= \left(\frac{en}{r}\right)^{r}\end{aligned}}$$

$$\frac{n!}{r!\,(n-r)!}$$

$$\frac{n!}{n_1!\,n_2!\,\ldots\,n_k!}$$

Sterling's formula

$$\underbrace{n!}_{} \approx \underbrace{\boxed{\sqrt{2\pi}}\; \underline{n}^{n+\frac{1}{2}} \cdot \underline{e}^{-n}}_{} \longrightarrow 1$$

$$n! \approx \sqrt{2\pi}\; n^{n+\frac{1}{2}}\, e^{-n}$$

$$\swarrow$$

$$\log(n!) =$$

$$\int_{k-1}^{k} \log \underline{x}\, dx < \underbrace{\log}_{} k \leq \underbrace{\int_{k}^{k+1} \log x\, dx}_{}$$

$$\sum_{k=1}^{n} \longrightarrow \int_{0}^{n} \log x\, dx \leq \log n!$$

$$\left[n! \simeq \left[\sqrt{2\pi}\; n^{n+\frac{1}{2}}\; e^{-n} \right] \right.$$

$$\log(n!) \simeq (\log\sqrt{2\pi}) + \left(n+\tfrac{1}{2}\right)\log n - n$$

$$\left[\int \log x \, dx = \underline{x \log x - x} \; \underline{+ C}\right]$$

$$\int_0^1 \log x \, dx < \underline{\log 1} < \int_1^2 \log x \, dx$$

$$\int_1^2 \log' x \, dx < \underline{\log 2} < \int_2^3 \log x \, dx$$

$$\vdots$$

$$\int_{n-1}^{n} \log x \, dx < \log n < \int_{n}^{n+1} \log x \, dx$$

$$\int_{0}^{n} \log x \, dx < \log(n!) < \int_{1}^{n+1} \log x \, dx$$

$$\left[x \log x - x\right]_{0}^{n} = n \log n - n \quad ✓$$

$\wedge|$

$\log(n!)$

$\wedge|$

$$\left[x \log x - x\right]_{1}^{n+1} = n+1 \log(n+1) - \quad ✓$$

$$-n$$

$$\frac{1}{2}\left[n \log n + (n+1) \log \overset{n+1}{n} - n - n \right]$$

$\log n!$ $\qquad$ $\left(n + \frac{1}{2}\right) \log n$

$$n \log n - n \quad \Big| \quad (n+1)\log(n+1) - n$$

$$\downarrow$$

$$\log n! \simeq \left(n+\frac{1}{2}\right)\log n - n$$

$$d_n = \underbrace{\log(n!) - \left(n+\frac{1}{2}\right)\log n + n}_{\to c}$$

$$d_1, d_2, \text{- - - - - - - - -} \boxed{d_n}$$

as $n \to \infty$

$$\boxed{d_n \longrightarrow \underline{\underline{c}}}$$

$$e^{LHS} = e^{RHS}$$

$$n! = \boxed{e^{c}}\, n^{n+\frac{1}{2}}\, e^{-n}$$

$$\boxed{\sqrt{2\pi}}\ \checkmark$$

$$\log n! =$$

$$c + \left(n+\tfrac{1}{2}\right)\log n$$

$$-n$$

$$\binom{n}{n\alpha}$$

where $\alpha < 1$

is a constant

$$\binom{n}{0} + \binom{n}{1} \cdots \leq \binom{n}{k}$$

–Stasys Jukna

Extremal Combinatorics

Concrete mathematics

[Knuth, Graham,
Patashnik]

Chapter 5

"r"

positive integer

$r^{\underline{k}}$

$\underline{r}^{\underline{k}}$

$r(r-1)(r-2)\ldots(r-k+1)$

$r = 5$

$r^{\underline{3}}$

$5 \cdot 4 \cdot 3$

$r = -3 \qquad r^{\underline{4}} = (\underline{-3})(-4)(-5)(-6)$

$r = \frac{1}{2} \qquad r^{\underline{3}}$

$\downarrow$

$= \frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right)$

falling factor power

$${}_nP_k = n^{\underline{k}}$$

$$= n(n-1)(n-2)\cdot(n-k+1)$$

$k \leq n$

$$n^{\underline{n}} = n!$$

$$n^{\underline{n+1}} = n(n-1)(n-2)\cdots 1\cdot 0$$

$$= 0$$

$$n^{\underline{1}} = n \qquad \underline{k \geq 0}$$

$$\boxed{n^{\underline{0}} = 1} \qquad \boxed{r^{\underline{0}} = 1}$$

$$\underline{\boxed{r^{\underline{-1}} = 0}}$$

rising factorial power

$$\boxed{r^{\overline{k}} = r(r+1)(r+2)\cdots(r+k-1)}$$

$$n^{\overline{1}} = n = n^{\underline{1}} \qquad \boxed{n^{\overline{0}} = 1}$$

$$1^{\overline{n}} = 1.\ 2.\ 3\cdots\ n = n!$$

$$n^{\underline{n}} = n! = 1^{\overline{n}}$$

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

---

upper index

any complex number

$$\binom{\boxed{r}}{k} = \begin{cases} \dfrac{r^{\underline{k}}}{k!}, & \text{integer } k \geqslant 0 \\ \underline{0} & \text{integer } \underline{k < 0} \end{cases}$$

lower index

any integer

$$(1+x)^{r \swarrow = -1} = \boxed{\sum_{k} \underbrace{\binom{\overset{-1}{r}}{k}} x^{k}}$$

$$(1+x)^{n}$$

$$\binom{\underline{\underline{n}}}{\boxed{k}}^{\swarrow} = \binom{n}{\boxed{n-k}}^{\swarrow}$$

$$\boxed{\binom{r}{k} = \binom{r}{r-k}}$$

r is also
an
integer

$$1 \nearrow \binom{-1}{0} = \binom{-1}{-1} = 0 \; ?$$

$$\frac{(-1)^{\underline{0}}}{0!} = \frac{1}{1} = 1$$

$$\xcancel{\binom{n}{n}} = 1 \qquad n = -1 \quad \binom{-1}{-1} = 0$$

r is an integer

$r = -1$ $\qquad k = 0$ ✓

$$\overset{0}{\curvearrowright}\binom{\boxed{-1}}{\boxed{k}^{-10}} = \binom{-1}{\underbrace{-1-k^{-10}}_{"9"}} \qquad k$$

$\underline{k < 0}$

$$\boxed{\binom{r}{k}} = \binom{r}{\underbrace{\overset{\checkmark}{r}-k}}\binom{-r}{9} = \frac{-1\cdot-2\cdot\ldots\cdot-9}{9!}$$

$$\boxed{=(-1)^9\frac{9!}{9!}}$$

$$\binom{\overset{0}{r}}{k} = \binom{r}{\boxed{r-k}}$$

$\boxed{0 \le k \le r}$

$k < 0$

$k > r$

$$\boxed{\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}}$$

$\underline{k \neq 0}$

$$\binom{r}{k} = \frac{r^{\underline{k}}}{k!} = \frac{r}{k}\boxed{\frac{r^{\underline{k-1}}}{(k-1)!}} = \underbrace{\frac{r}{k}\binom{r-1}{k-1}}_{}$$

$k=0$

$$\boxed{k\binom{r}{k}} = \boxed{r\binom{r-1}{k-1}}$$

$$0 = 0$$

$$k\binom{r}{k} = r\binom{r-1}{k-1},$$ all integer k

$$(r-k)\binom{r}{k} = r\binom{r-1}{k}$$ all integer k

$$(r-k)\binom{r}{k} = (r-k)\binom{r}{r-k} = r\binom{r-1}{r-k-1} = r\binom{r-1}{k}$$

all r
all k

$$r-1-(r-k-1) = k$$

$$\binom{r}{k} = \frac{\textcircled{r}^{\underline{k}}\ ✓}{k!\ ✓} = \frac{r(r-1)\cdots(r-k+1)\ ✓}{k!\ ✓}$$

$$\xrightarrow{}\ \underline{\text{k th degree}}$$

$$\binom{r}{3} = \frac{r^{\underline{3}}}{3!} = \frac{r(r-1)(r-2)}{6}$$

$$\binom{r}{k} \qquad = \frac{(r^2-r)(r-2)}{6}$$

$$= \frac{(r^3 - 3r^2 + 2r)}{6}$$

$$\left[\boxed{(x-k)\binom{x}{k} - x\binom{x-1}{k} = 0}\right]$$

for all int k

when r is a positive integer

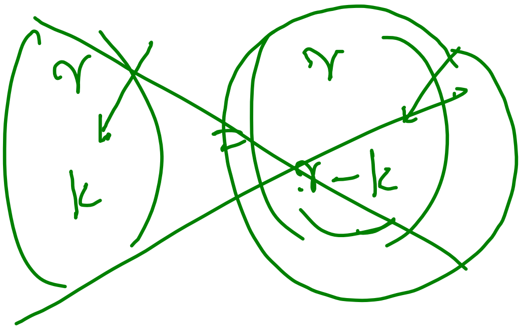
$r^{\underline{k}}$

$$\underbrace{\left[0x^{k+1} + 0x^{k} + 0x^{k-1} + \cdots\right]} = 0$$

only +ve integer r

"any r"

"polynomial argument"



$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$$

0 0 0

all integers "k"

$$\boxed{k > 0}$$

$1 = 1 + 0$ ✓ when $k=0$

$$\boxed{\binom{r}{k} = \binom{r-1}{k-1} + \binom{r-1}{k}}$$ ✓

$0 \le k \le r\ldots$

$$\boxed{\binom{r}{k} - \binom{r-1}{k-1} - \binom{r-1}{k}} = 0$$

$$(x+y)^{r\in\mathbb{R}} = \boxed{\sum_{k} \binom{r}{k} x^{k} y^{r-k}}$$

"r"

$\boxed{0 \le k \le r}$

$r = 1/2$

$k \geq r+1$

$$\binom{r}{k} = \frac{r^{\underline{k}}}{0!}$$

↓

$$\left|\frac{x}{y}\right| < 1$$

$$y^r \times \left(1+\frac{x}{y}\right)^r = \left[\sum_k \binom{r}{k} x^k y^{r-k}\right]$$

$$(y+x)^r =$$

$$\boxed{\left|\frac{x}{y}\right|<1}$$

$$z = x/y$$

$$(1+z)^r = \sum_k \binom{r}{k} z^k$$

$$|z|<1$$

$$f(z) = (1+z)^r$$

$$f(z) = \frac{f(0)}{0!} z^0 + \frac{f'(0)}{1!} z^1 + \frac{f''(0)}{2!} z^2 + \cdots + \frac{f^{(k)}(0)}{k!} z^k + \cdots$$

$$(1+z)^r$$

$$\frac{r^{\underline{1}}}{1!} = \binom{r}{1}$$

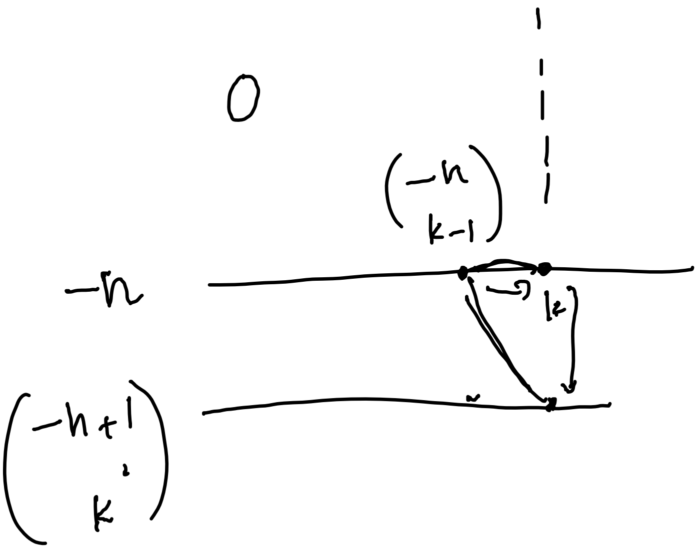
$$\frac{r^{\underline{k}}}{k!} = \binom{r}{k}$$

$$\boxed{|z| < 1}$$

row#							
-3							
-2							
-1	1	-1	1	-1	1	-1	...
0	1						
1	1	1					
2	1	2	1				

$$\binom{n}{r}\checkmark = \binom{n-1}{r-1}\checkmark + \binom{n-1}{\underline{r}}\checkmark$$

$$\underbrace{\binom{-1}{k}} = \boxed{\binom{-2}{k}} + \binom{-2}{k-1}\checkmark$$



$$\begin{pmatrix} \overset{n}{0} \\ 0 \end{pmatrix}_{k} = \begin{pmatrix} \overset{n-1}{-1} \\ 0 \end{pmatrix}_{k} + \begin{pmatrix} \overset{n-1}{-1} \\ -1 \end{pmatrix}_{k-1}$$

$$\downarrow$$

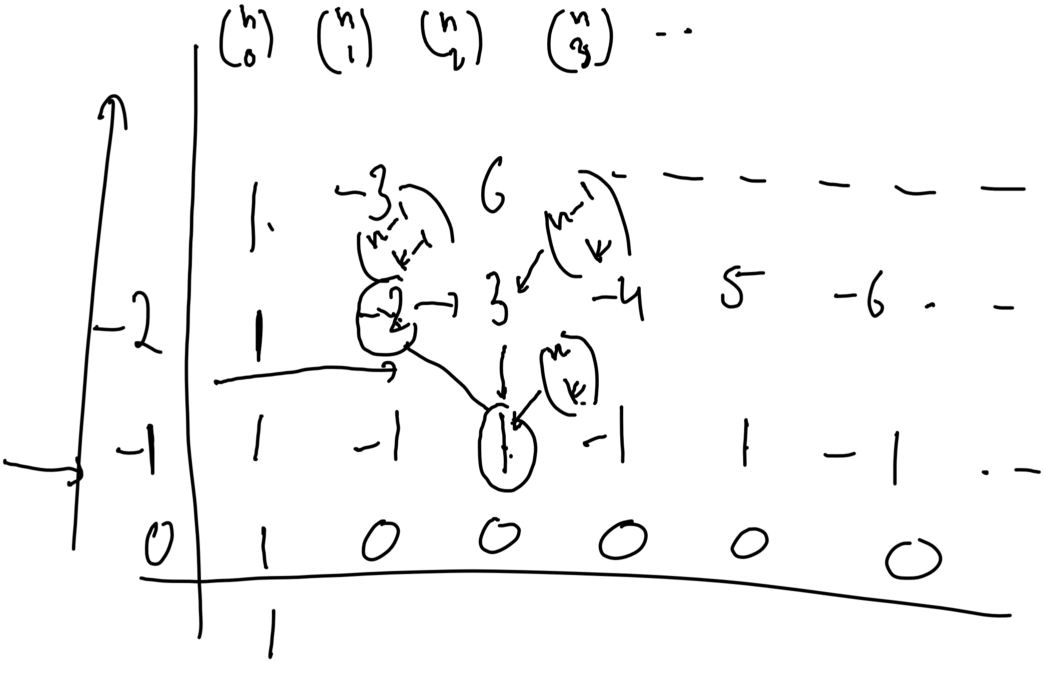
$$1 = 1 + \begin{pmatrix} -1 \\ -1 \end{pmatrix}^{0}$$

$$\frac{\gamma^{0}}{0!} = 1$$

$$\binom{\overset{n}{0}}{\underset{k}{1}} = \underbrace{\binom{\overset{n-1}{-1}}{1}} + \binom{-1}{0}$$

$$0 = \underline{-1} + 1$$

$$0-1$$



	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	..		
	1	-3	6	- -	- -	- -	- -
-2	1	-2	3	-4	5	-6	. .
-1	1	-1	1	-1	1	-1	. .
0	1	0	0	0	0	0	
	1						

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$$\binom{r}{k} = (-1)^{k} \binom{k-r-1}{k}$$

$$\binom{5}{4} = \boxed{(-1)^{4}} \binom{-2}{4}$$

$$\binom{-3}{5} = (-1)^5 \binom{5+3-1}{5} = -\binom{7}{5}$$

$$\binom{\boxed{r}}{k} = (-1)^k \binom{\underline{k-r}-1}{k}$$

$$\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}$$

$$= \boxed{(-1)^k} \frac{(-r)(1-r)(2-r)\cdots(k-1-r)}{k!}$$

$$= (-1)^k \frac{(k-r-1)^{\underline{k}}}{k!} = (-1)^k \binom{k-r-1}{k}$$

$$\binom{r}{k} = (-1)^{\overset{0}{k}} \binom{\overset{0}{k}-r-1}{k_0}$$

all integers k

$1 = 1$ ✓

$$\boxed{k>0}$$

$k-r-1$

$k-r-2$

$k-r-3$

$\vdots$

$\cancel{k}-r\cancel{-1}-\cancel{k}\cancel{+r}$

$-r$

$$\binom{r}{k} = (-1)^{k} \binom{k-r-1}{k}$$

$$\binom{r}{k} = \frac{r^{\underline{k}}}{k!} = \frac{r(r-1)\cdots(r-k+1)}{k!}$$

$$= (-1)^{k \xleftarrow{\cancel{k!}}} \frac{[-r][1-r][2-r]\cdots[k-r-1]}{k!}$$

$$=(-1)^{k}\frac{(k-r-1)^{\underline{k}}}{k!}$$

$$=(-1)^{k}\binom{k-r-1}{k}$$

$r=3$

$k=5$

$$3^{\underline{5}}=[3\cdot 2\cdot 1\cdot 0\cdot -1]=0$$

$$=(-1)^{5}=[(-3)(-2)(-1)(0)\,1]$$

$$1^{\underline{5}}\leftarrow 1\times 0\times -1\times -2\times -3$$

$$=0$$

$$7^{\underline{4}} = (7 \cdot 6 \cdot 5 \cdot 4)$$

$$= (-1)^4 (-7)(-6)(-5)(-4)$$

$$= (-1)^4 (-4)(-5)(-6)(-7)$$

$$\underline{= (-1)^4 (-4)^{\overline{4}}}$$

$r = 5$ $\quad$ $k > r$ $\quad$ $\binom{5}{7} = 0$

$k = 7$

$$\binom{r}{k} = (-1)^k \binom{k - \overset{-1}{\underset{s}{r}} - \cancel{1}}{k}$$

k column

$-1 \; - - \; 1 \; -1 \; + \; \cdots + 1 \; \cdots$

0 –

1 –

$\vdots$

k

$$\binom{-1}{k} = (-1)^k \binom{k \cancel{+1-1}}{k}$$

$\binom{0}{k}$

$\binom{0}{1}$

$\boxed{\binom{k}{k}}$

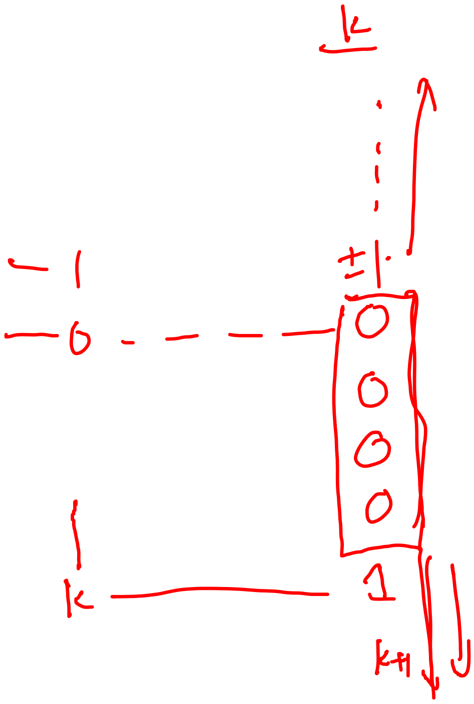
$\vdots$

$$\binom{k\,\cancel{\ }^{n}}{k} = (-1)^k \binom{k - k - 1}{k}$$

$$= (-1)^k \binom{-1}{k}$$

$$\binom{k+1}{k} = (-1)^k \binom{k-(\cancel{k}+1)-1}{k}$$

$$\binom{k}{k} = (-1)^k \binom{-1}{k} = (-1)^k \binom{-2}{k}$$



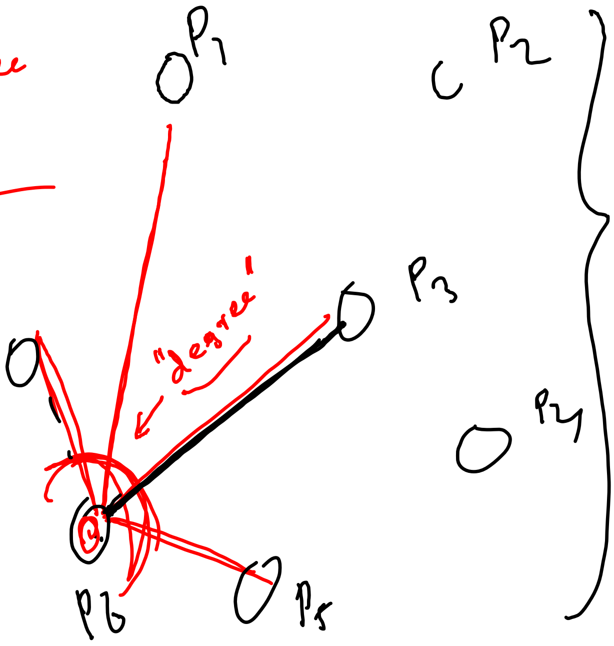
$$\sum_{i=1}^{m} r_i = \sum_{i=1}^{n} c_i$$

$$\begin{array}{c|cccccc|c}
 & 1 & 2 \cdots & & - & & n & \\
\hline
1 & 0 & 1 & 0 & 0 & 1 & 0 & r_1 \\
2 & 1 & 1 & 1 & 0 & 0 & \cdots & r_2 \\
3 & 0 & 0 & 1 & 1 & 1 & \cdots & r_3 \\
\vdots & & & & & & & \vdots \\
i & & & & & & & \\
\vdots & \cdots & & & & & & \vdots \\
m & & & & & & & r_m \\
\hline
 & c_1 & c_2 & c_3 & \cdots c_i & \cdots & c_n &
\end{array}$$

\# odd degree vertices

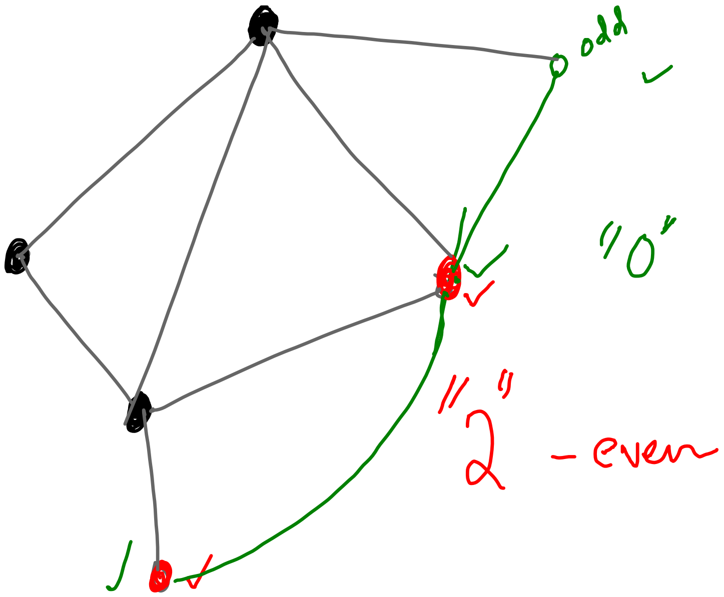
↓

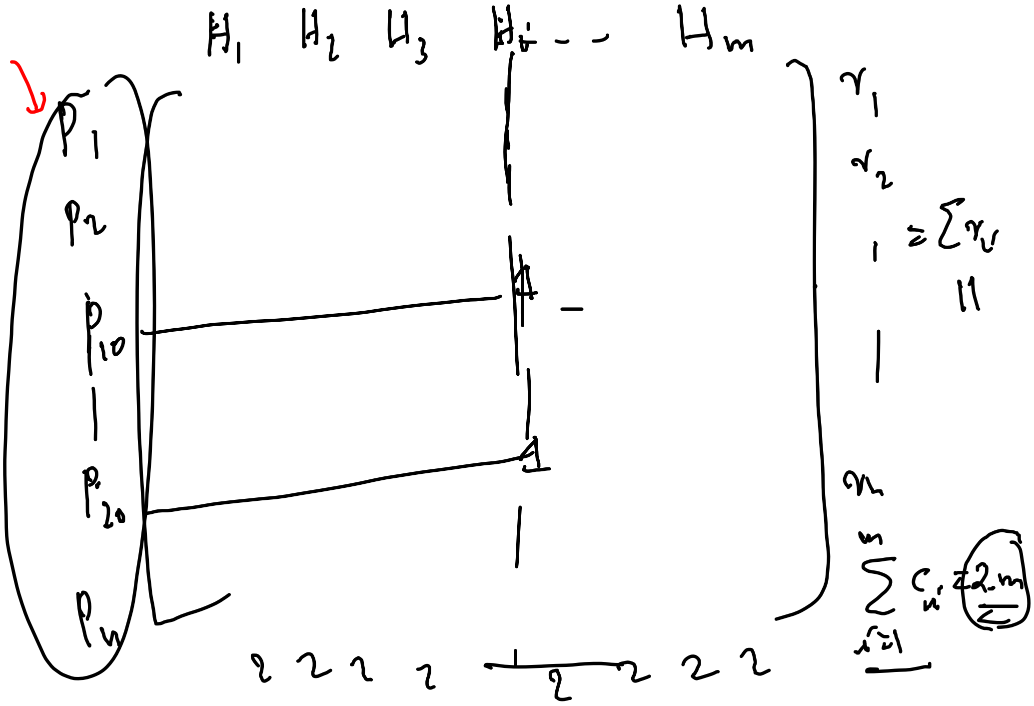
odd



"10"

\# of peoples who shook hands an odd number


odd
✓
"0"
✓
✓
"2" - even
1
✓


H_1 H_2 H_3 H_i - - - H_m
p_1
p_2
p_{10}
p_{20}
p_n
1
1
r_1
r_2
$= \sum r_i$
n
$\sum_{i=1}^{m} c_i = 2m$
2 2 2 2 2 2 2 2

$H \rightarrow$ "2"

$m \rightarrow \boxed{2m}$

$$\boxed{\sum r_i = 2m}$$ ✓ even

\# persons

even

$$\overset{2m}{\boxed{\sum r_i}} = \underbrace{\overset{k}{\text{odd}} \times \overset{\text{even}}{\text{odd}}}_{x} + \underline{\overset{\text{even}}{(n-k)} \times \text{even}}$$

$= \underline{\text{odd}}$

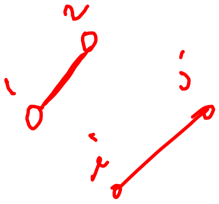
$U = \{1, 2, ..\}$ $F = \{ S_1, S_2, . . , S_m \}$

$$\boxed{\text{each } S_i \subseteq U}$$

$$U = \underbrace{\{1, 2, 3, .., n\}}$$

hyper edges

$\{1, 2\}$



$S_1, S_2, . . . S_m$ ✓

$\{i, j\}$

m-sets

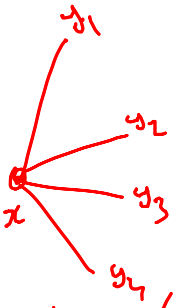
each $|S_i| = k$

k-uniform hyper graph

$\mathcal{F}$

$x \in U$

$d(x) =$

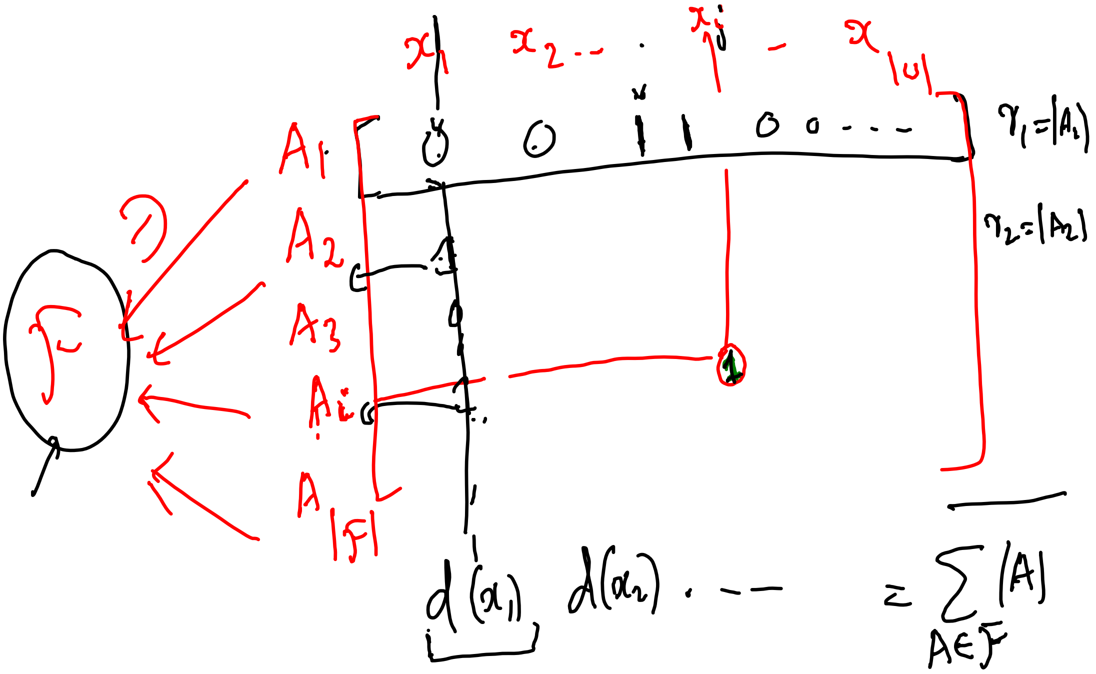


$\mathcal{F} = \{ \{x, y_1\}, \{x, y_2\},$
$\{x, y_3\} \{x, y_4\}$
$---$

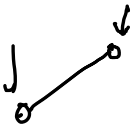
$$d(x) = \left| \{ S \in \mathcal{F} : \quad x \in S \} \right|$$

$|A|$

2-uniform
k-uniform



$$\sum_{i=1}^{|U|} C_i = \sum_{x \in U} d(x) = \sum_{i=1}^{|F|} r_i = \sum_{A \in F} |A|$$

Graph 

2-uniform hypergraph

$$\left(|A_1| = |A_2| = - - = |A_{|F|}| = \text{"}2\text{"} \right)$$

$$\sum_{A\in\mathcal{F}} |A| = \underline{2\,|\mathcal{F}|} \swarrow \underline{\underline{m}}$$

$$\sum d(\overset{\checkmark}{x}) = \underline{\overset{\swarrow}{2}\,|F|}$$

$l \le k \le n \longrightarrow [n]$

↳ 2^n

l-element

$\binom{n}{l}$

$l \le k$

k-element

$\binom{n}{k}$ $\boxed{|S| = k}$

$k=2$

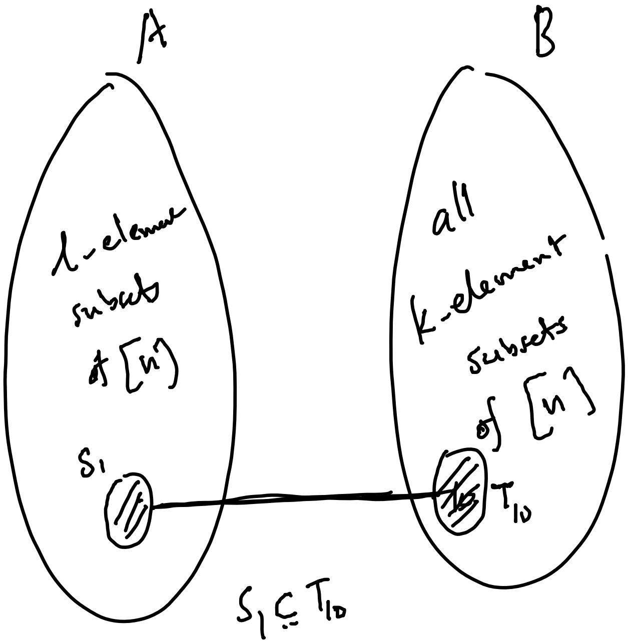
$[n]=[10]=\{1, \underline{2, 3, 4, 5, 6, 7, 8, 9, 10}\}$

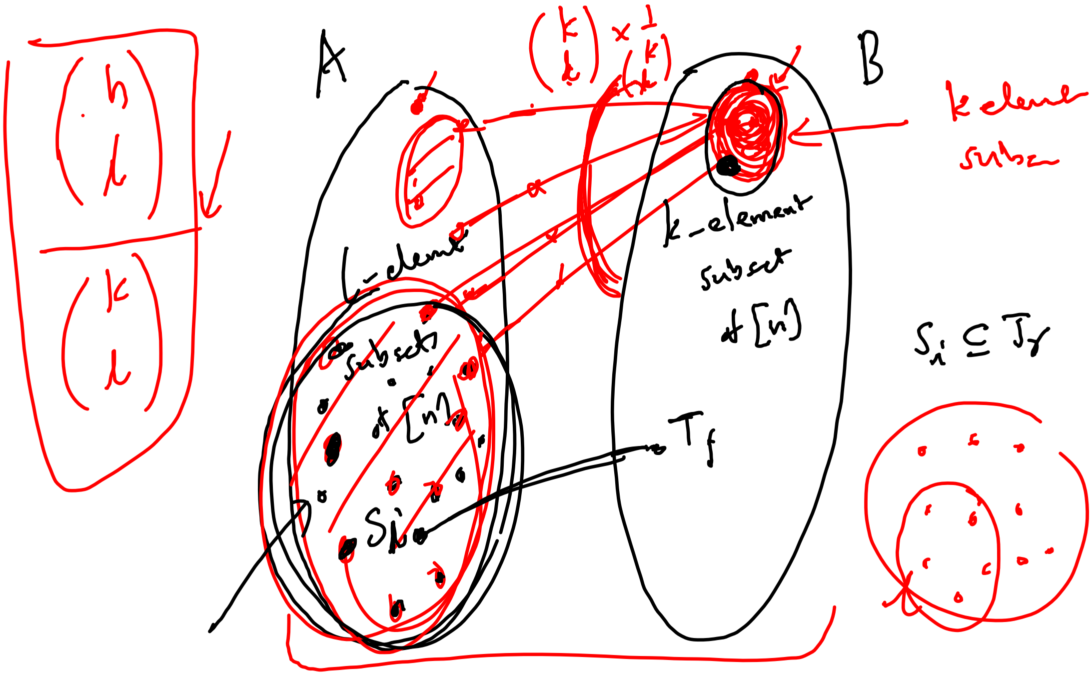
"8" "2" $k=2$ $\binom{10}{2}$

$l=1$

$\underbrace{\text{How many } l\text{-element}}_{9\text{-people}}$

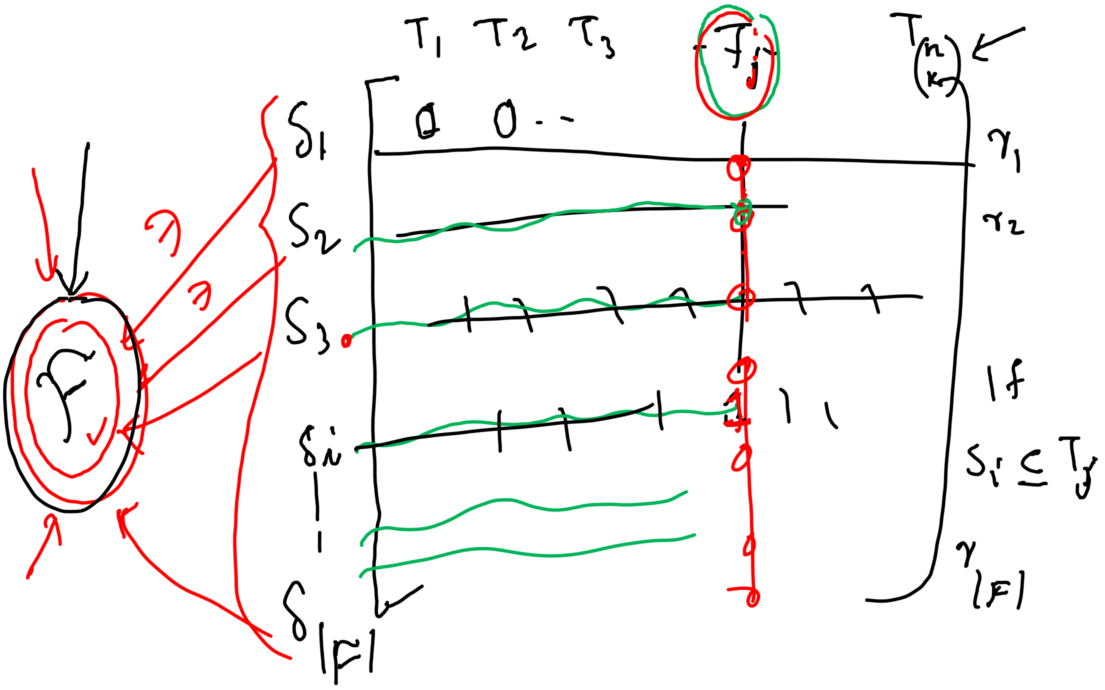
$\underline{2\text{-members}}$


A
B
l-element subsets of [n]
all k-element subsets of [n]
S_1
T_{10}
$S_1 \subseteq T_{10}$


$\frac{\binom{n}{l}}{\binom{k}{l}}$
A
B
$\binom{k}{l} \times \frac{1}{\binom{k}{l}}$
k-element subset
l-element subsets of [n]
k-element subset of [n]
S_i
T_f
$S_i \subseteq T_f$

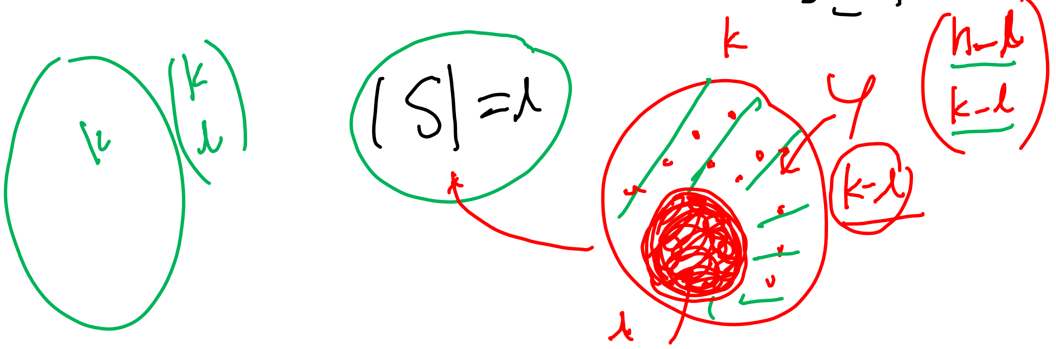
$\mathcal{F}$ is a family of ℓ-element subsets having the required property

$\mathcal{F} \subseteq \binom{[n]}{\ell}$, we want to show $\underline{|\mathcal{F}| \geq \binom{n}{\ell} \Big/ \binom{k}{\ell}}$



$$\sum \gamma_i = \sum_{i=1}^{|\mathcal{F}|} \binom{n-l}{k-l} = |\mathcal{F}| \binom{n-l}{k-l} = \sum c_i =$$

$$Y \qquad S \subseteq Y$$



$$1 \times \binom{n}{k} \quad \boxed{\sum_{i=1}^{\binom{n}{k}} c_i \le \binom{k}{\underline{l}} \times \binom{n}{k}} = \frac{\cancel{k!}}{l!\,(k-l)!} \; \frac{n!}{\cancel{k!}\,(n-k)!}$$

$$\sum$$

$$|\mathcal{F}| \binom{n-l}{k-l} \ge \binom{n}{k}$$

~~$|F| \leq \frac{\binom{k}{l}\binom{n}{k}}{\binom{n-l}{k-l}} = \frac{\cancel{k!}}{l!\cancel{(k-l)!}} \cdot \frac{n!}{\cancel{k!}\cancel{(n-k)!}} \cdot \frac{(n-l)!}{\cancel{(k-l)!}\cancel{(n-k)!}}$~~

$$\frac{n!}{l!\cancel{(k-l)!}\cancel{(n-k)!}} \quad \frac{\cancel{(k-l)!}\cancel{(n-k)!}}{(n-l)!}$$

$$= \frac{n!}{l!(n-l)!} = \boxed{\binom{n}{l}}$$

$$\binom{n}{k} \times 1 \leq \sum c_i \leq \cancel{\binom{n}{k}\binom{k}{l}}$$

$$|\mathcal{F}| \geq -$$

$$\sum r_i = |\mathcal{F}| \binom{n-l}{k-l} = \sum c_i \geq \underline{\bigcirc}$$

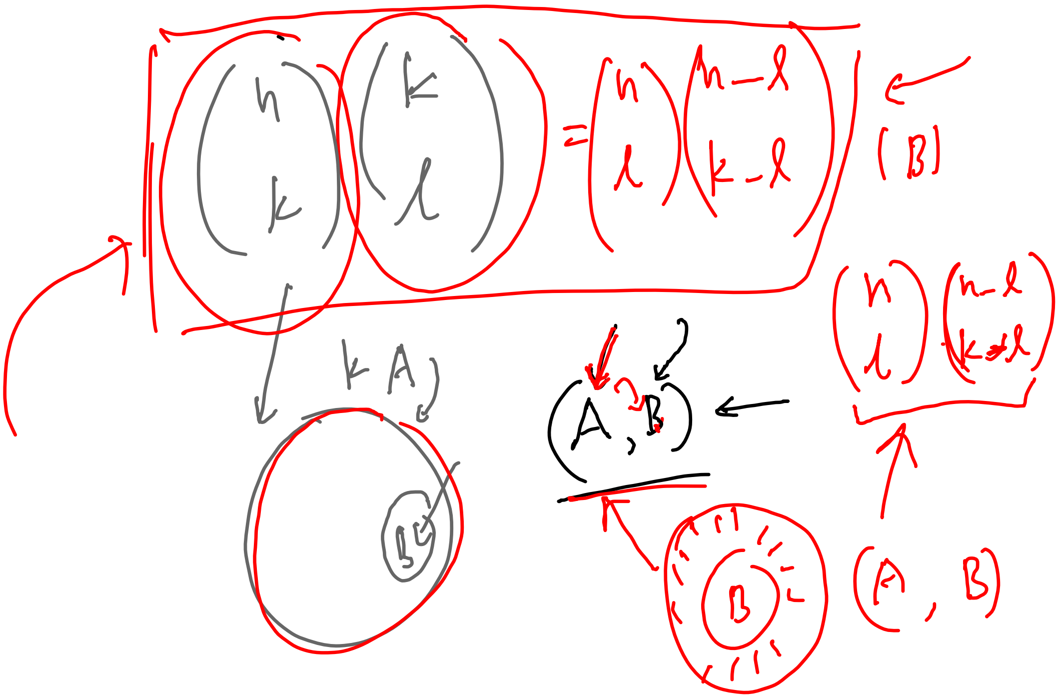
$$|\mathcal{F}| \binom{n-l}{k-l} \geq \binom{n}{k}$$

$$\boxed{|\mathcal{F}| \geq \frac{\binom{n}{k}}{\binom{n-l}{k-l}} = \frac{\binom{n}{l}}{\binom{k}{l}}}$$

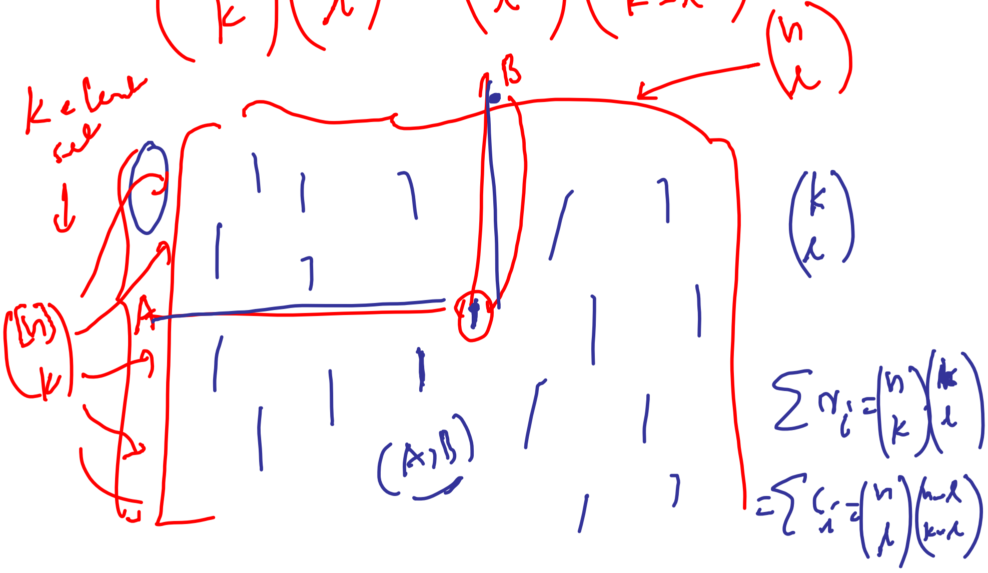
$$\frac{\binom{n}{k}}{\binom{n-l}{k-l}} = \frac{\binom{n}{l}}{\binom{k}{l}} \quad ?$$

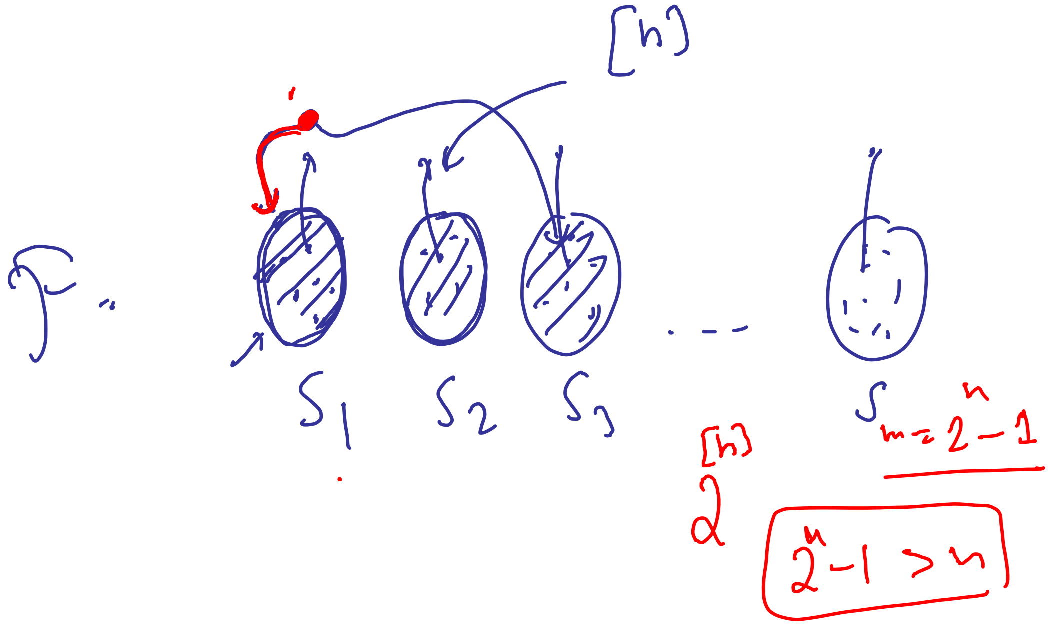
$$\binom{n}{k}\binom{k}{l} = \binom{n}{l}\binom{n-l}{k-l}$$

$$\left\{ \begin{array}{l} \frac{\cancel{n!}}{\cancel{k!}\,\cancel{(n-k)!}} \frac{\cancel{k!}}{\cancel{l!}\,\cancel{(k-l)!}} \overset{?}{=} \frac{\cancel{n!}}{\cancel{l!}\,\cancel{(n-l)!}} \frac{\cancel{(n-l)!}}{\cancel{(k-l)!}\,\cancel{(n-k)!}} \\ \\ \qquad\qquad 1 = 1 \end{array} \right.$$

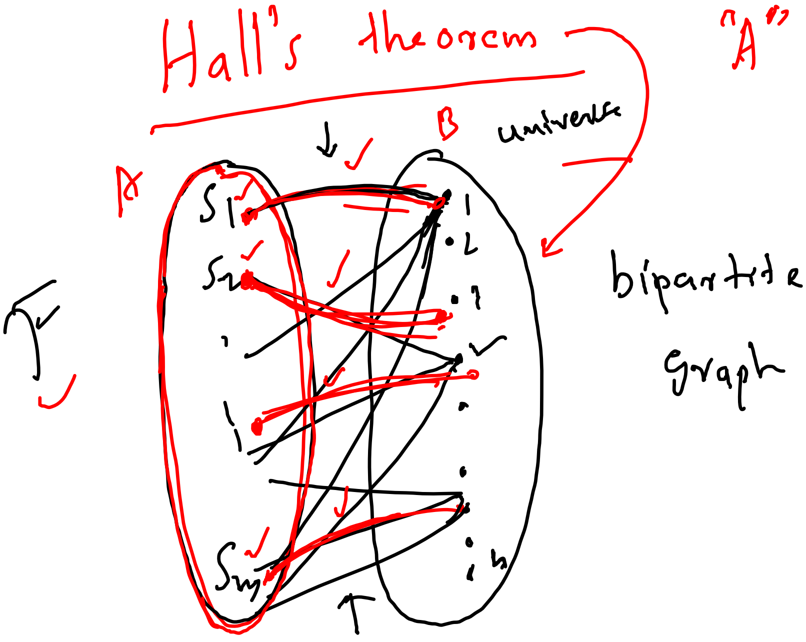


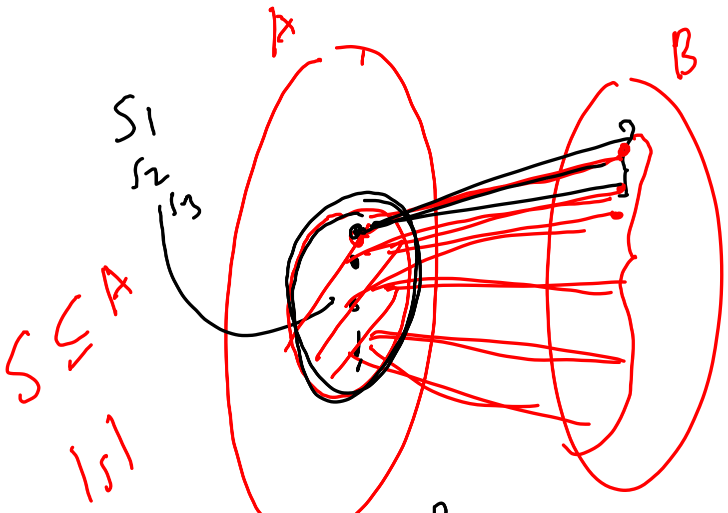
$$\binom{n}{k}\binom{k}{l} = \binom{n}{l}\binom{n-l}{k-l}$$





Hall's theorem





$|S_1 \cup S_2 \cup S_3| \geq 3$ Hall's

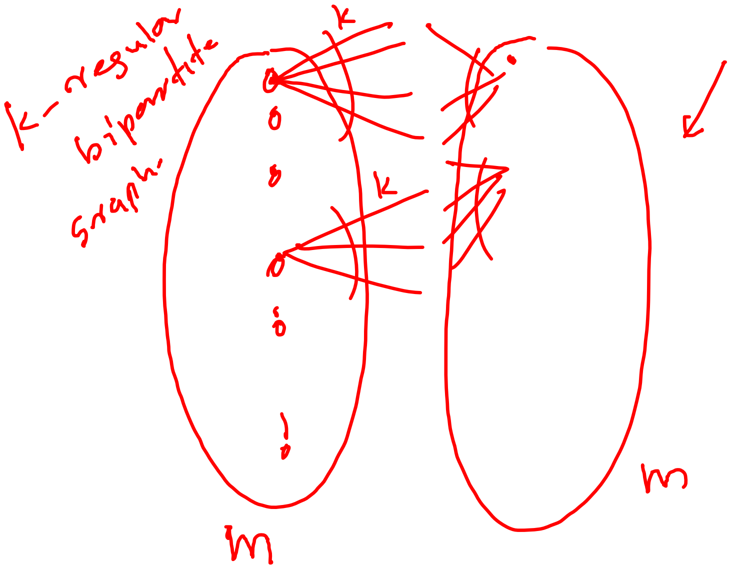
$\underbrace{|N(S)| \geq |S|}$ ✓

Hall's condition is satisfied

$S_{10}, S_{13}, S_{16}, S_{25}$

a, b, c ✓ a ✓ b ~~c~~

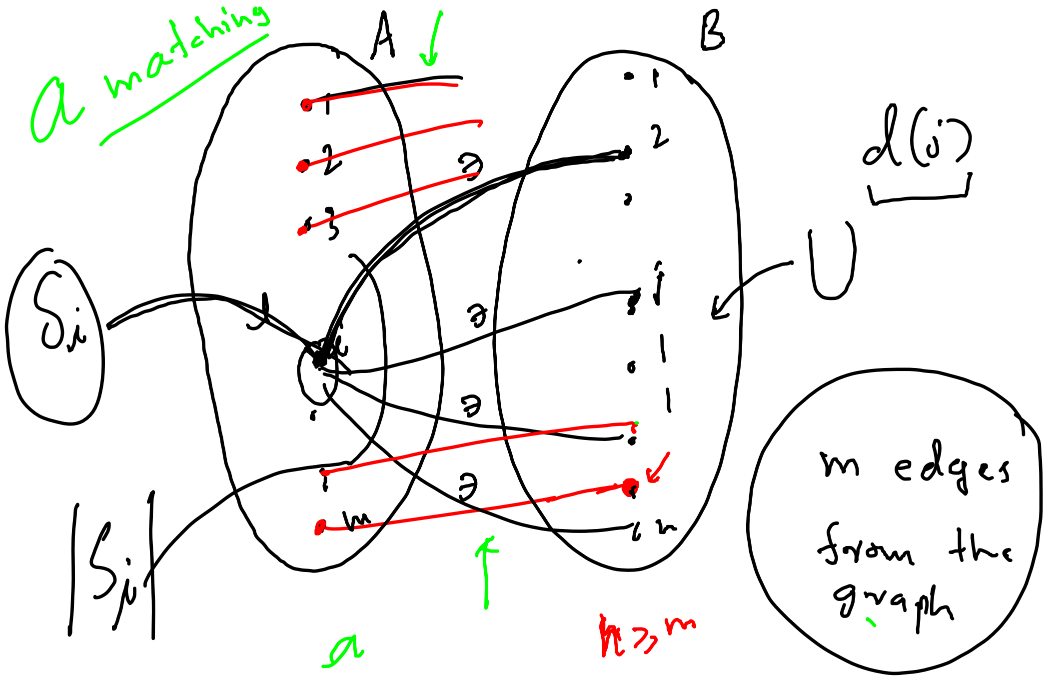
$$\left\{ |S_{10} \cup S_{13} \cup S_{16} \cup S_{25}| \right\} \geq 4$$



System of distinct representatives.

$$|U| = n \quad U = [n] \to \boxed{m \le n}$$

$$\mathcal{F} = \{\underbrace{\overset{x}{S_1}, S_2, \ldots, S_m}\}$$



Hall's theorem

Hall's condition

" for every $S \subseteq A$, $\boxed{|\underbrace{N(S)}_{}| \geq |S|}$ "

$\underline{\forall S \subseteq A, \ \underline{|N(S)| \geq |S|}}$ Hall's condition

then, $\exists$ a matching of A

$$|\boxed{N(S)}| = \left| \bigcup_{S_i \in S} S_i \right| \geq |\underbrace{S}|$$

k - subsets from $\mathcal{F}$

$\underline{S_1, --- S_m}$

in $\binom{m}{k}$ ✓

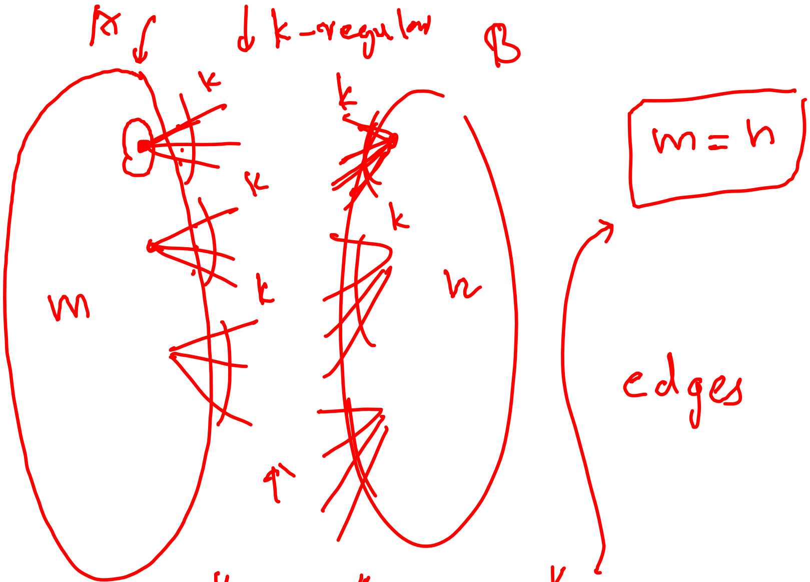
2^m - subsets of

necessary

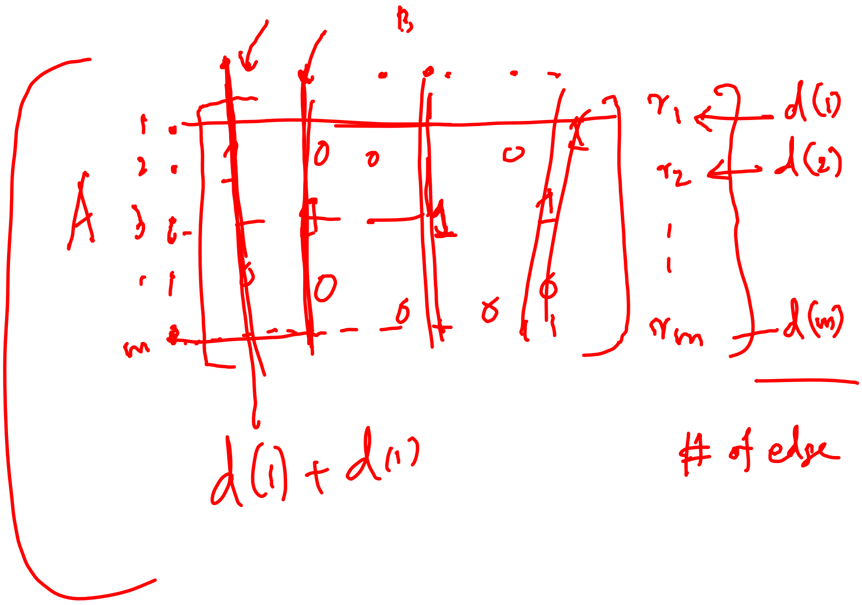
$$\left\{ \left| \underset{\underset{x_1 \checkmark}{\uparrow}}{S_{i_1}} \cup \underset{\underset{x_2 \checkmark}{\uparrow}}{S_{i_2}} \cup \underset{\underset{x_3 \checkmark}{\uparrow}}{S_{i_3}} \cup \dots \underset{\underset{x_k \checkmark}{\uparrow}}{S_{i_k}} \right| < k \right\}$$

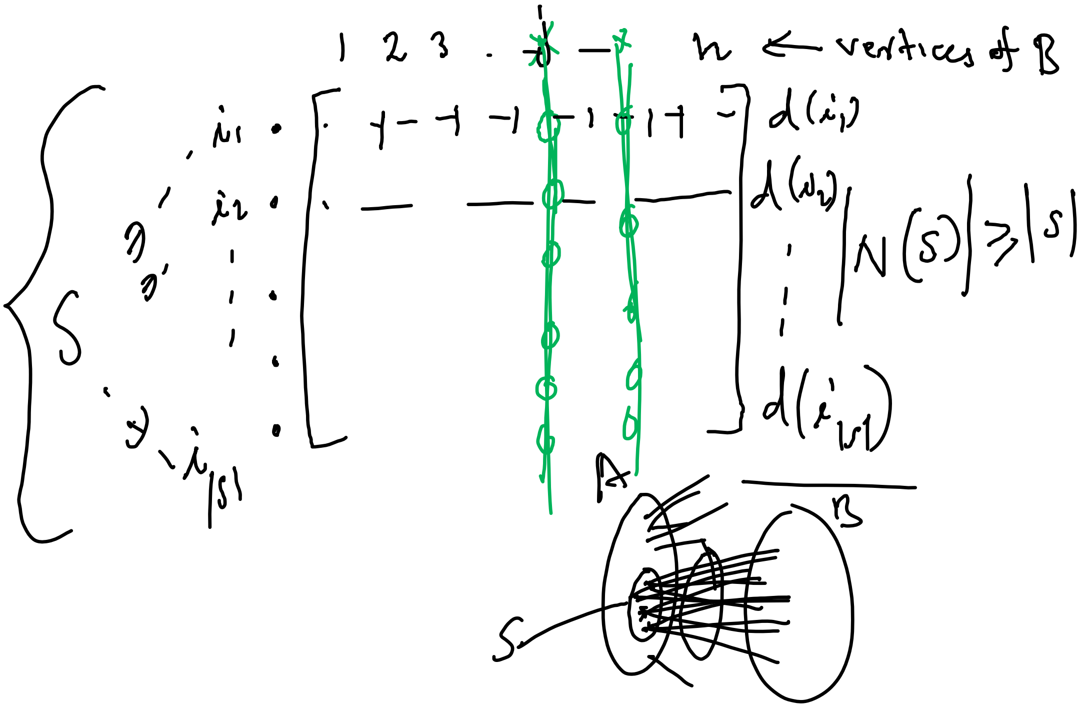
$\dots$ }

k

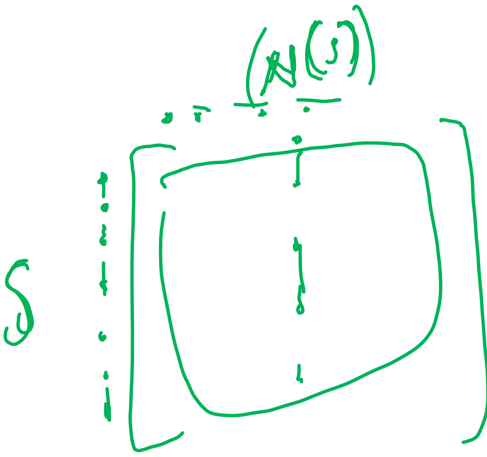


"$m \times \cancel{k} = n \times \cancel{k}$


B
A
1
2
3
i
m
0
1
r_1
r_2
r_m
d(1)
d(2)
d(m)
d(1) + d(1)
# of edge


1 2 3 . j — n ← vertices of B
S
i_1
i_2
$i_{|S|}$
j
d(i_1)
d(i_2)
d($i_{|S|}$)
|N(S)| ≥ |S|
A
B
S

$|N(S)| \cdot 1 \leq$ # edges in G



$|N(S)| \times \cancel{k}$

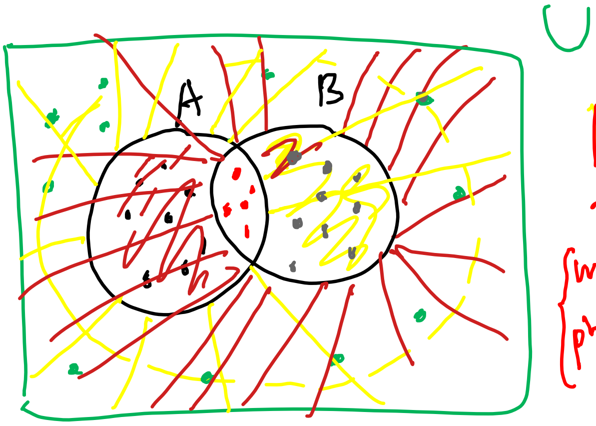
$\geq \boxed{|S| \cancel{k}}$

$\boxed{|N(S)| \geq |S|}$

$\overline{A \cup B}$

$\uparrow$

$\underbrace{\bar{A} \cap \bar{B}}$

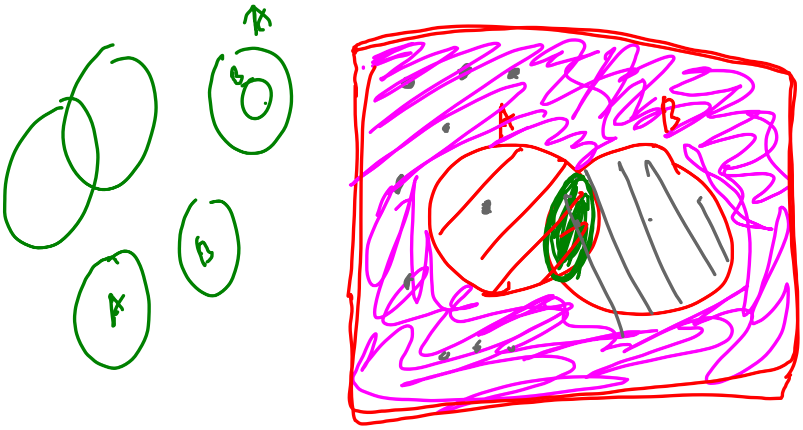


100

{mathematics
physics

A↙

B↙

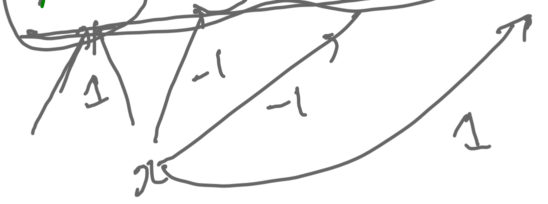


|B|

$+|A\cap B|$

$$\bar{A}\cap\bar{B} = \overline{A\cup B} = |U| - |A| - |B|$$

$$|\overline{A\cup B}| = \boxed{|U|} - \overset{0}{|A|} - \overset{0}{|B|} + \overset{0}{|A\cap B|}$$

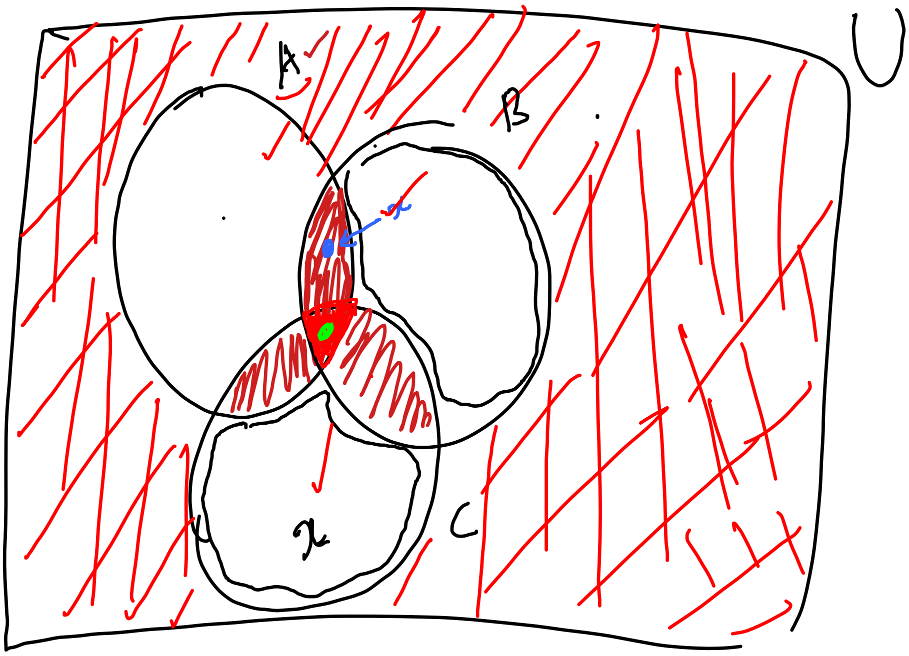


$\overline{A\cup B}$

$1-1-1\;1=\underline{\underline{0}}$

$\underline{1-1=0}$ $\quad x\in B-A$

$x \in \underline{A \cap B}$



$$\underline{\bar{A}\cap\bar{B}\cap\bar{C}} \;\; \Big[\; \underbrace{\overset{1}{|U|}}_{1} - \underbrace{\overset{-1}{|A|}}_{-1} - \underbrace{\overset{0}{|B|}}_{-1} - \underbrace{\overset{0}{|C|}}_{-1}$$

$$+ \underbrace{\overset{+1}{|A\cap B|}} + \underbrace{\overset{+1\;0}{|B\cap C|}}$$

$$\underline{1 - 1 + 1 - 1 = 0} \quad + \underbrace{\overset{+1\;0}{|C\cap A|}} - \overset{1\;0}{|A\cap B\cap C|} \;\Big]$$

$$\underbrace{1 - \binom{3}{1} + \binom{3}{2} - \binom{3}{3}} = 0$$

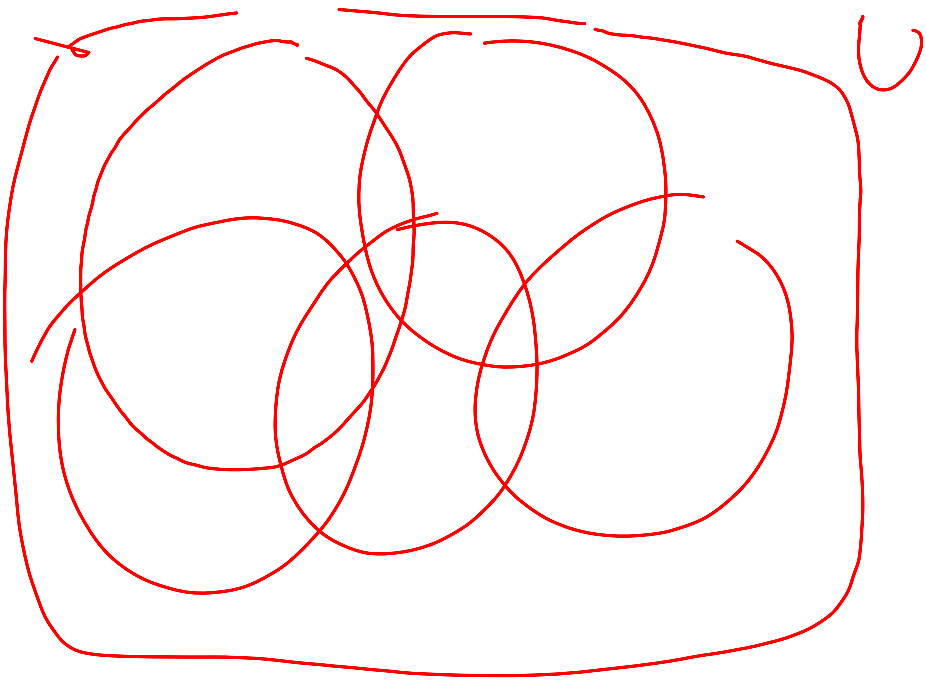
$$\underline{1 - 3 + 3 - 1}$$

$$x = 0 \quad \boxed{(1+x)^3 = \binom{3}{0} \binom{3}{1} + \binom{3}{2} - \binom{3}{3}}$$

$A_1, A_2, A_3, \ldots A_n$

$A_i \subseteq U$

$\bigcap \overline{A}_i = \overline{(A_1 \cup A_2 \cup \cdots \cup A_n)}$



$$\left|\bigcap \overline{A_i}\right| = |U| - \sum_{i=1}^{n} |A_i| + \sum_{i,j} |A_i \cap A_j| - \sum_{i,j,k} |A_i \cap A_j \cap A_k| + \dots + (-1)^{n} |A_1 \cap \dots \cap A_n|$$

$$|U| - \underset{\checkmark}{|A_1|} - |A_2| - |A_j| \dots - |A_k|$$

$$+ \sum_{i,j} |A_i \cap A_j| - \sum \qquad \binom{[k]}{3}$$

$$\underline{x \in A_1 \cap A_2 \cap A_3 \cap A_4}$$

$$\left|\bigcap \overline{A}_i\right| = \left|\overline{A_1 \cup \dots \cup A_k}\right|$$

$$\left[\begin{array}{l} x \in U - (A_1 \cup A_2 \cup \dots \cup A_k) \\ \underline{x \in \bigcap \overline{A}_i} \end{array}\right.$$

$$\boxed{\overset{1}{|U|}} - \cancel{\sum \overset{0}{|A_i|}} + \cancel{\sum_{i,j} \overset{0}{|A_i \cap A_j|}}$$

$$- \cancel{\sum_{i,j,k} \overset{0}{|A_i \cap A_j \cap A_k|}}$$

$$+ \cdots\cdots$$

$$x \in \bigcap \overline{A_i}$$

$$x \in \overset{\checkmark}{A_{i_1}} \cap \overset{\checkmark}{A_{i_2}} \cap \overset{\checkmark}{A_{i_3}} \cdots \overset{\checkmark}{A_{i_t}}$$

$x \notin A_{i_j}$ when $j \notin [1, t]$

$$\left|\overset{\overset{x}{\swarrow}\,1}{\bigcup}\right| - \underset{\binom{t}{1}}{\overset{\overset{x}{\swarrow}\;t}{\sum}} |A_i| + \underset{i,j \;\; \binom{t}{2}}{\overset{\binom{t}{2}}{\sum}} \underline{\underline{|A_i \cap A_j|}}$$

$$\nearrow$$

$$\binom{t}{2} \quad x \begin{cases} A_{i_1} \\ A_{i_2} \\ \vdots \\ A_{i_t} \end{cases}$$

$$1 - \binom{t}{1} + \binom{t}{2} - \binom{t}{3} + \binom{t}{4} + \cdots (-1)^t \binom{t}{t} = 0$$

$$1 = \binom{t}{0}$$

$$\binom{t}{3} \qquad \sum_{i,j,k} (\quad) \qquad \#$$

$$\boxed{A_{i_1} A_{i_2} \cdots A_{i_t}}$$

$$\boxed{\binom{t}{0} - \binom{t}{1} + \binom{t}{2} + \cdots (-1)^t \binom{t}{t} = 0}$$

$$\left| \bigcap \overline{A_i} \right| =$$

$$\boxed{x \in U} \qquad \cancel{|U|}$$

$$\underline{A_1 \cdots A_t}$$

$\underline{5, 10, 15, 20, 25}$ $\quad \overset{\checkmark}{5}, \overset{\checkmark}{6}, \overset{\checkmark}{8}$ $\quad \left\lfloor \frac{26}{8} \right\rfloor = 5$

$\underline{1 \leq \ldots n \leq 1000}$

$\boxed{P_1(n) = n \text{ is divisible by } 5}$

$\boxed{\frac{1000}{5} = 200}$

$\underline{1 \ldots \overset{\checkmark}{\boxed{5}} \ldots \textcircled{10}} \ldots \textcircled{15} \ldots$ $\quad \left\lfloor \frac{13}{5} \right\rfloor = 2$

$10/5 = 2$

$A_1 = \{ n : n \text{ has Property } P_1 \text{ and } 1 \leq n \leq 1000 \}$

$A_2 = \{ n : P_2(n) \,\&\, 1 \leq n \leq 100 \}$

~~100~~

$|A_2| = \left\lfloor \frac{\overset{166}{1000}}{6} \right\rfloor = \underline{166}$

$|A_3| = \left\lfloor \frac{1000}{8} \right\rfloor = \underline{125} = \{ n : \underline{P_3(n)} \,\&\, 1 \leq n \leq 1000 \}$

$1 \le n \le 1000$

which donot have properties P_1, P_2 & P_3

$$U = [1000] = \{1, 2, \cdots, 1000\}$$

$$\underline{\boxed{\overline{A_1}} \cap \boxed{\overline{A_2}} \cap \boxed{\overline{A_3}}}$$

$$= U - (A_1 \cup A_2 \cup A_3)$$

$$= |U| - \sum_{i=1}^{3} A_i + \underline{\sum_{i,j} |A_i \cap A_j|}$$

$$\downarrow \qquad - \underline{|A_1 \cap A_2 \cap A_3|}$$

$$= 1000 - (200 + 166 + 125) + (33 + 25 + 41)$$

$$- 8$$

$$|A_1 \cap A_2|$$

$$\left\lfloor \frac{100\cancel{0}}{3\cancel{0}} \right\rfloor = 33 \qquad \boxed{\begin{array}{c} 5 \mid n \\ \text{and } 6 \mid n \\ 30 \mid n \end{array}}$$

$\left\lfloor \frac{1000}{40} \right\rfloor = 25$

5 and 8 $\leftarrow$ n

$40 \mid n$

$40 \mid n$

$\left\lfloor \frac{1000}{120} \right\rfloor = 8$

✓ ✓ ✓

5, 6, 8 $\rightarrow$ $5 \times 3 \times 8$

✓ ✓ ✓

$\boxed{120}$

$|A_1 \cap A_2| = 33$ ✓

✓ ✓

$6 \wedge 8$

$|A_1 \cap A_3| = 25$ ✓

$2 \times 3 \,;\; 2^3$

$|A_2 \cap A_3| = \left\lfloor \frac{\cancel{1000}^{\cancel{250}\,\cancel{125}\;41}}{\cancel{24}\,\cancel{6}\,3} \right\rfloor$

$= 41$ ✓

$2^3 \times 3 = 24$

$24 \mid n$

$[1000]$

$\nearrow$

$|U| = \underline{1000}$

MATH IS FUN '9'

$\downarrow$

M$\boxed{\text{IS}}$AFTHUN 9!

$\underline{\text{IFSUN '}\boxed{\text{MATH}}\text{'}}$

"9!" permutations of 9 letters

M' IS → U

P_1 → "MATH" appears consecutive in the permutation

M' IS FUN,

A_1 6!

P_2 "IS" appears consecutive ← '9'

$|A_2| = 8!$

F'

P_3 'FUN' "

$|A_3| \longrightarrow 7!$

$$\underbrace{A_1}_{\downarrow} \quad \underbrace{A_2}_{\downarrow} \quad \underbrace{A_3}_{\downarrow}$$

IS "FUN"

$$\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$$

$$= U - (A_1 \cup A_2 \cup A_3)$$

$$|U| - \underbrace{\sum_{i=1}^{3} |A_i|} + \boxed{\sum_{i,j} |\underline{A_i \cap A_j}|}$$

$$- \underbrace{|A_1 \cap A_2 \cap A_3|}$$

$$\swarrow$$

$$9! - (6! + 8! + 7!)$$

$5!$ $|A_1 \cap A_2|$

M' I'

$A_1 \cap A_3$ MATH IS

$(4!)$ M' F' IS $4!$

$9-6+2=5$

MATH FUN $9-7+2=4$

$|A_2 \cap A_3|$ 2

IS FUN

I' F'

$9-5+2=6$

$A_1 \cap \underset{\smile}{A_2} \cap A_3$

✓

M' I' F'

MATH IS FUN

$3!$

$$|\bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_k|$$

$$= |U - (A_1 \cup A_2 \cup .. A_k)|$$

$$= |U| - \sum \quad + \sum$$

$$\overline{A_1} \cap \dots \overline{A_k} = \overline{(A_1 \cup \dots \cup A_k)}$$

$$|A_1 \cup \dots \cup A_k| = |U| - \left(\bigcap \overline{A_i}\right)$$

$$= |U| - \left(|U| - \sum A_i + \sum_{i,j} A_i \cap A_j \dots\right)$$

$$\underline{|A_1 \cup A_2 \cup \dots \cup A_k|}$$

$$= \sum_{i=1}^{k} A_i - \sum_{i,j} \underbrace{A_i \cap A_j}_{}$$

$$+ \dots + (-1)^{\underbrace{k+1}_{}} (A_1 \cap \dots \cap A_k)$$

$$|A_1| = |A_2| = \dots = |A_k| = \alpha_1$$

$$|A_1 \cap A_2| = |A_1 \cap A_3| = \dots$$

$$= |A_i \cap A_j| = \dots = \alpha_2$$

$$|A_i \cap A_j \cap A_k| = \alpha_3$$

$$\vdots$$

$$|A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}| = \alpha_k$$

$$\left|\overline{A_1 \cup A_2 \cup \cdots \cup A_k}\right|$$

$$= \bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_k = \overbrace{|U|}^{\alpha_0} - \overbrace{\sum_{i=1}^{k} |A_i|}^{\binom{k}{1}\alpha_1}$$

$$+ \underbrace{\sum_{i,j} \overset{\binom{k}{2}\alpha_2}{|A_i \cap A_j|}}_{} - \underbrace{\overset{\alpha_3}{\sum_{i,j,k} |A_i \cap A_j \cap A_k|}}_{\binom{k}{3}\alpha_3}$$

$$\boxed{= \alpha_0 - \underbrace{\binom{k}{1}}\alpha_1 + \underbrace{\binom{k}{2}}\alpha_2 - \underbrace{\binom{k}{3}}\alpha_3 \cdots (-1)^k \alpha_k}$$

0 99999

10 }
99 }

100 }
999 }

1000 }
9999 }

0-9

00000 00100

10 10 10 10 10

0... 9

10^5

10^5

0, - - - , 99999 }

$\boxed{1,00,000}$

$0 \cdots\cdots \quad \underbrace{99999}^{\times}$

100238 ✓ $\quad 2, 3, 8$

$\underline{23008}$

$\underline{21038} \quad 98\cancel{7}65$

$U = [99999] \cup \{0\}$

$|U| = 10^5$

$P_1 \rightarrow$ the number does not contain $\underbrace{2}$

$P_1(n)$

99999 ✓

87654 ✓

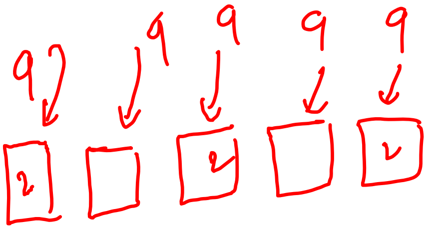
$$\begin{tikzcd}
A_1 & P_1 \arrow[l] \\
A_2 & P_2 \arrow[l] \\
A_3 & P_3 \arrow[l]
\end{tikzcd}$$

$P_2(n) \rightarrow$ n does not have the digit 3

$P_3(n)$ – n does not have the digit 8

$$|A_1| = 9^5$$

"2"



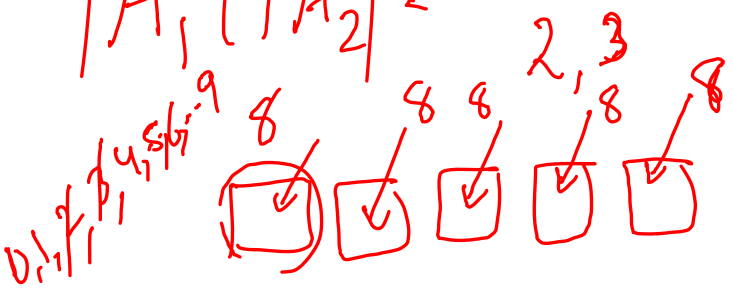
0, 1, 3, 4, .. 9

9^5

$$\left\{ \underbrace{|A_1|} = \underbrace{|A_2|} = \underbrace{|A_3|} = \underbrace{9}^5 \right.$$

$0 \cdots \underline{99{,}999}$

$|A_1 \cap A_2|$ ←

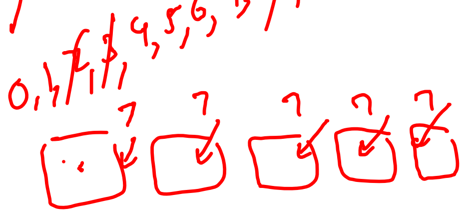


8^5

$$|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3|$$

$$= 8^5$$

$$|A_1 \cap A_2 \cap A_3| = 7^5$$



2 3 8

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |U| - \binom{k}{1}\alpha_1 + \binom{k}{2}\alpha_2 + \binom{k}{3}\alpha_3$$

α_0 ?

$$10^5 - 3\cdot 9^5 + \binom{3}{2}\cdot 8^5 + \binom{3}{3}7^5$$

2, and 3, and 8

$$S = \{3\cdot a,\ 4\cdot b,\ 4\cdot c\}$$

10 10 10

$$\{a,a,a,b,b,b,b,c,c,c,c\}$$

10 combin

∞ times

r- combinatu "n" types

$$\binom{r+n-1}{r}$$

$$a\,a\,a\,a\,a\,a\,a\,a\,a\,a$$

$$\left(\underline{a\,a\,\overset{\alpha}{a}\,a}\,b\,b\,b\ldots\right)$$

$$\underline{r=10}$$

$$\binom{\overset{10}{r}+\overset{3}{n}-1}{r}=\binom{10+3-1}{10}$$

$$=\underline{\underline{\binom{12}{10}}}$$

$S = \{3 \cdot a, 4b, 4c\}$

$x \quad y$

$\binom{r+n-1}{r}$

$\underline{10\text{-combinations}}$ ✓ $\binom{r+n-1}{r}$

$\boxed{|U| = \binom{12}{10}}$

$\nearrow [\,P_1(c) \rightarrow$ the 10-combination c has atleast 4 $\underline{a\text{'s}}$

$\nearrow [\,P_2$

$\nearrow [\,P_3$

$A_1 \leftarrow$

$A_2 \leftarrow$

$A_3 \leftarrow$

$\rightsquigarrow P_2(c)$ – the 10-combination c has at least 5-bs

2

$P_3(c)$ – 10-combination c has at least 5 c-s

$$\underbrace{\left|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}\right|}_{} \leftarrow \underbrace{\begin{Bmatrix} 3, a \\ 4 . b \\ 4 . c \end{Bmatrix}}_{}$$

$$= \overset{\binom{12}{10}}{|U|} - \sum_{i=1}^{3} |A_i| + \sum_{i,j} |A_i \cap A_j| - \sum_{i,j,k} |A_i \cap A_j \cap A_k|$$

S

Set of 10-combinations of a, b, c with at least 4 a's

↕

4 10-comb

Set of 6-combinations of a, b, c

$$\binom{\overset{6}{r}+\overset{3}{n}-1}{\underset{6}{r}} = \binom{8}{6}$$

$$|A_1| = \binom{8}{6}$$

A_2 # 10-combinations such that there are at least 5 bs mit

$A_2' \to$ 5-combinations of a, b, c

$$|A_2| = \binom{7}{5}$$

$\underline{|A_3| = \binom{7}{5}}$

A_3 (5 c)

A_3'

$\underline{\text{5-combination}}$

$|A_3| = |A_3'|$

$r = 5 \quad n = 3$

$|A_1 \cap A_2|$?

$|B| = 3 \quad B = |A_1 \cap A_2| \rightarrow$ at least 4 a s in it

and at least 5 b s

~~a, b, c~~

$|B'| \quad \binom{\overset{1}{r} + \overset{3}{n} - 1}{r} = \binom{3}{1} = 3$

$\underbrace{\text{1-combinations of } a, b, c}_{\{a\},\ \{b\}\ \{c\}}$

$$|\underbrace{A_1}_{} \cap \underbrace{A_3}_{}| = \underline{3}$$

"9"

$$|\underbrace{A_2 \cap A_3}_{}| = \underline{1}$$

5 - b's

5 - c's

10's

$$\boxed{|A_1 \cap A_2 \cap A_3| = 0}$$

"10" - combi

$$\left\{\begin{array}{l} 4 \text{ a's} \\ 5 \text{ b's} \\ 5 \text{ c's} \end{array}\right.$$

14

$$S = \{ \underset{\overset{\uparrow}{1}}{n_1} \cdot a_1, \ \underset{\overset{\uparrow}{1}}{n_2} \cdot a_2, \ \dots, \ \underset{\overset{\uparrow}{1}}{n_k} \cdot a_k \}$$

r-combination $\Big\}$ $\dfrac{\text{if each } n_i \geq r}{\dbinom{r+k-1}{r}}$

$$\binom{k}{r} \ ✓$$

$\underline{k=3}$

$$\boxed{x_1 + x_2 + x_3 + \cdots + x_k = r}$$

for each x_i, $\boxed{0 \leq x_i \leq n_i}$ ✓

✓

$$\boxed{0 \le x_i \le n_i}$$

We use inclusion-exclusion

$1 \le x_1 \le 5$

$x_1 \longrightarrow y_1 = x_1 - 1$

$0 \le y_1 \le 4$

$-2 \le x_2 \le 4$

$y_2 = x_2 + 2$

$0 \le y_2 \le 6$

$3 \le x_4 \le 9$

$y_4 = x_4 - 3$

$0 \le y_4 \le 6$

$x_1 - 1$ $\quad x_2 + 2$ $\quad x_3$ $\quad x_{\cancel{3}\,4} - 3$

$\boxed{x_1 + x_2 + x_3 + x_4 = 18}$ — (I)

$\boxed{y_1 + y_2 + y_3 + y_4 = \underline{\underline{16}}}$ — (II) n

$$\boxed{y_1+y_2+y_3+y_4=16} \leftarrow \text{II}$$

n_1, n_2, n_3, n_4

$$0 \le y_1 \le 4,\quad 0 \le y_2 \le 6,\quad 0 \le y_3 \le 5,\quad 0 \le y_4 \le 6$$

$$\binom{r+k-1}{r} = \binom{16+4-1}{16} = \binom{19}{16}$$

$$|U| = \binom{19}{16}$$

$\sum y_i = 16$

$\sum z_i = -2$

$A_1 \leftarrow P_1 \rightarrow y_1 \ge 5$ ✓ $\qquad z_1 = y_1 - 5$

$A_2 \leftarrow P_2 \rightarrow y_2 \ge 7$ ✓ $\qquad z_2 = y_2 - 7$

$A_3 \quad P_3 \rightarrow y_3 \ge 6$ ✓ $\qquad z_3 = y_3 - 6$

$A_4 \quad P_4 \rightarrow y_4 \ge 7 \qquad z_4 = y_4$

$\binom{14}{11} = |A_1|$ $\quad y_1 \geq 5 \quad$ $\boxed{0 \leq y_2, y_3, y_4}$

$\boxed{0 \leq y_1, y_2, y_3, y_4}$

$z_1 = y_1 - 5$

$0 \leq z_1$

$$\underset{y_1-5}{z_1} + \underset{y_2}{z_2} + \underset{y_3}{z_3} + \underset{y_4}{z_4} = 11$$

$\boxed{z_1, z_2, z_3, z_4 \geq 0}$ $\quad \binom{11+4-1}{11} = \binom{14}{11}$

$|A_2| = \binom{12}{9}$ $\quad \underbrace{y_2 \geq 7}$

$$y_1 + y_2 + y_3 + y_4 = 16$$

$$z_1 + z_2 + z_3 + z_4 = 9$$

$z_2 = y_2 - 7$

$\binom{r+k-1}{r} = \binom{9+4-1}{9}$

$\boxed{z_i \geq 0}$

$$\binom{7}{4} \quad |A_i \cap A_j| \qquad y_1 \geq 5$$

$$y_2 \geq 7$$

$$\| $$

$$\binom{4+4-1}{4} = |A_1 \cap A_2| \; ?$$

$$y_1 + y_2 + y_3 + y_4 = 16$$

$$y_1 - 5 \leftarrow z_1 + \underbrace{z_2}_{y_2 - 7} + z_3 + z_4 = 4$$

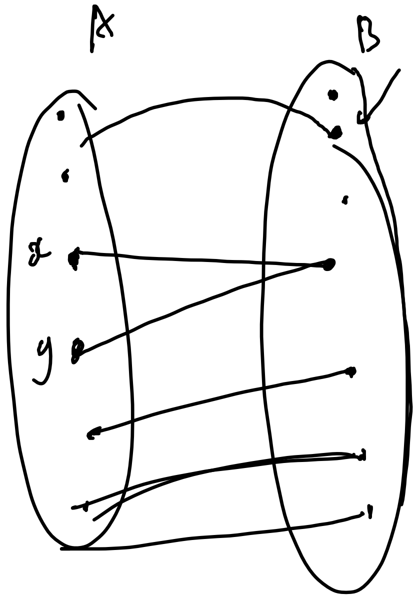
$$A_1 \cap A_2 \cap A_3$$

$$\boxed{|U| - \sum |A_i| + \sum_{i,j} |A_i \cap A_j|}$$

$$- \sum_{i,j,k} \cancel{|A_i \cap A_j \cap A_k|}$$

$$+ \sum \cancel{\;-\;}$$

$\underline{f(x) = f(y)}$



$\underline{x \in B}$

$x \notin \{f(y) : y \in A\}$

↙ into

$\boxed{|A| = |B|}$

Onto functions

$$\boxed{|A|} \geq |B|$$

$$\underline{|B| = \left|\{f(\underline{x}) : \underline{x \in A}\}\right|}$$

$$\leq |A|$$

$|B| = |A|$

$f: A \to B$

$|A| = m$

$|B| = n$

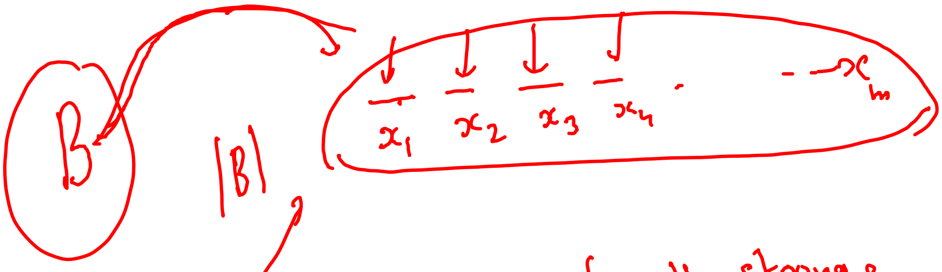
$\boxed{n^m}$

n^m $A = \{x_1 ✓, x_2 ✓, -- x_m ✓\}$

$f(x_1) = n$

$f(x_2) = k n$

$f(x_m)$



$|B|$

n

n^m

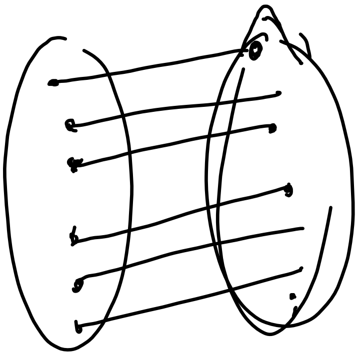
m-length strings

into functions

$$n(n-1)\cdots(n-m+1)\} \qquad \boxed{|B| \geq |A|} \checkmark$$

$$= n^{\underline{m}}$$

$$= \underline{{}^{n}P_{m}}$$



$f(x_1) \leftarrow n$

$f(x_2) \leftarrow n-1$

$f(x_3) \leftarrow n-2$

# of onto functions

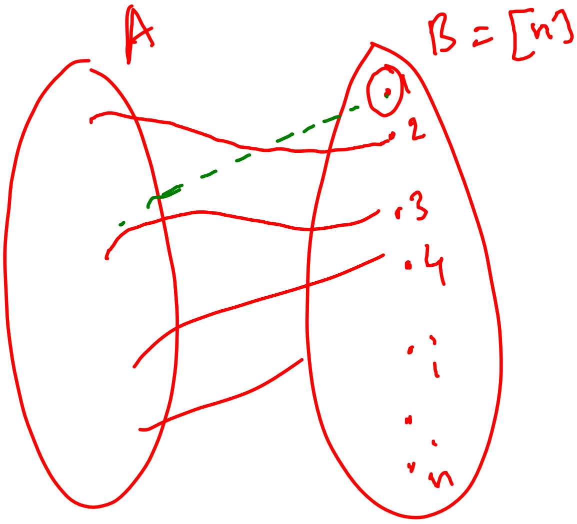
$$|A| = m \geq n = |B|$$

when $m < n$

onto functions from A to B is 0

U = set of all functions from A to B

$|U| = n^m$



$$\begin{cases} P_1 \rightarrow 1 \in B \text{ does not have any pre-image.} \\ P_2 \rightarrow 2 \in B \text{ does not have any preimage} \\ \vdots \\ P_i \longrightarrow i \in B \text{ does not have any pre-image} \\ P'_n \longrightarrow n \in B \quad '' \end{cases}$$

$$A_i \leftarrow$$

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}|$$

$$= |U| - \sum_{i=1}^{n} A_i + \sum_{i,j}^{n} |A_i \cap A_j| + \cdots + (-1)^n |A_1 \cdots A_n|$$

$$\boxed{\alpha = |A_i| = (n-1)^m}$$

$f : A \to B$

$$\alpha_2 = |A_{\underline{i}} \cap A_{\underline{j}}| = (n-2)^m$$

$f(x) = i$

$f(x) = j$

$\{ f : A \to \underbrace{B \setminus \{i, j\}}_{n-2}$

$$|A_i \cap A_j \cap A_k| = (n-3)^m$$

$$|A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_t}| = (n-t)^m$$

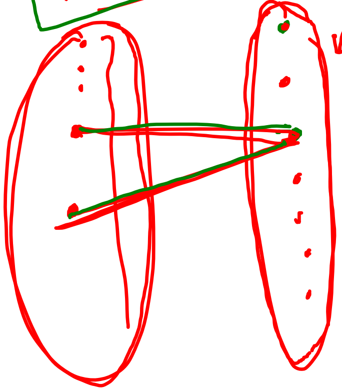
onto fns from A → B

$$\overset{\alpha_0}{|U|} - \binom{n}{1}\alpha_1 + \binom{n}{2}\alpha_2 - \binom{n}{3}\alpha_3 + \cdots + (-1)^n \binom{n}{n}\alpha_n$$

$$= n^m - n\cdot(n-1)^m + \binom{n}{2}(n-2)^m + \cdots (-1)\binom{n}{n}(n-n)^m$$

$n = m$ # of onto fns

$\boxed{m = n}$ = # of bijection ✓



$A \to B$

$n! = m!$

$$\begin{cases} \sum_{i=0}^{m}\binom{n}{i}(n-i)^{m}(-1)^{i} = n! \quad \underline{\text{If } n=m} \\ \underline{m<n} \\ \sum_{i=0}^{n}\binom{n}{i}(n-i)^{m}(-1)^{i} \quad = \underline{\underline{0}} \end{cases}$$

i, j are relatively prime

$\gcd(i,j)=1$

$\underline{\gcd(10, 21)}$

$= 1$

$\gcd(3, 7) = 1$

$\boxed{1 \ldots\ldots\ldots n-1}\ n$

$n \geq 2$

$\phi(n)$

$1\ \cancel{2}\ 3, \cancel{4}, 5$

$1, \cancel{3}\ 5$

$\phi(2) = 1$

$\phi(3) = 2$

$1, 2$

$\phi(4) = 2$

$\phi(5) = 4$

$\phi(6) = \cancel{3}\ 2$

$$\phi(p) = p-1$$

$1 \ldots\ldots p-1$

when $n = p_1^{e_1}\, p_2^{e_2} \cdots p_k^{e_k}$

$$\phi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)$$

p

[illegible]

$p_1 = p$

$e_1 = 1$

$k = 1$

$$\phi(n) = p^1 \times \left(1 - \frac{1}{p}\right)$$

$$= \frac{p\,(1 - p)}{p} = 1 - p$$

$$6 = 2^1 \cdot 3^1$$

$$\phi(6) = 6 \times \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right)$$

$$= 6 \times \frac{1}{2} \; \frac{2}{3}$$

$$= 2$$

① ~~2~~ ~~3~~ ~~4~~ ⑤ 6

"2"

$$U = [n-1] = \{1, 2, --, n-1\}$$

$$n = \underline{\underbrace{p_1^{e_1}}\ \underbrace{p_2^{e_2}} \cdots \underbrace{p_k^{e_k}}}$$

$m \in U$, $\to H_1 \to$ $\underline{p_1 \text{ devides } m}$

$\vdots$

$H_i \to p_i$ devide m

$$A_i =$$

$$\underline{\qquad\qquad\qquad\qquad\qquad\qquad}$$

$$n = p_1^{e_1}\, p_2^{e_2}\, p_3^{e_3} \cdots p_t^{e_t}$$

$$\boxed{\phi(n) = n \prod_{i=1}^{t} \left(1 - 1/p_i\right)}$$

$A_1 \subseteq U \leftarrow H_1(k) = p_1$ devides k ✓ $\quad U = [n]$ ✓ $\{1, \ldots, n\}$

$A_2 \subseteq U \quad H_2(k) = p_2$ devides $k \quad p_1, p_2, \ldots p_t$

$H_i(k) = p_i$ devides k

$A_t \subseteq U \quad H_t(k) = p_t$ devides $\quad 1 \leq k \leq n-1$

$$A_i = \{ k \in \underset{\checkmark}{[n]} : H_i(k) \text{ is true} \}$$

$$\left| \underbrace{\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \dots \cap \overline{A_t}} \right|$$

$$= \underbrace{|\overset{n}{\downarrow}\!U|} - \boxed{\sum |A_i|} + \underbrace{\sum_{i,j} |A_i \cap A_j| \checkmark}$$

$\ast$

1, 2, ⋯ n−1

$$\dots - \sum_{i,j,k} |A_i \cap A_j \cap A_k| \checkmark$$

$$+ \dots + (-1)^t |A_1 \cap \dots \cap A_t|$$

$$= n - \sum_{i=1}^{t} \frac{n}{p_i} + \sum_{i,j} \frac{n}{p_i p_j} - \sum_{i,j,k} \frac{n}{p_i p_j p_k} + \cdots + (-1)^{t} \frac{n}{p_1 p_2 \cdots p_t}$$

divisible

$\frac{n}{p_i}$ $\frac{n}{p_j}$ both by p_i and p_j

$\underbrace{p_i p_j}$ devides k

$$\frac{n}{(p_1 p_2 \cdots p_k)} \left[\overset{p_1 p_2 \cdots p_k}{1} - \sum_{i=1}^{t} \frac{1}{p_i} + \sum_{1 \le i, j \le t} \frac{1}{p_i p_j} - \sum_{i,j,k} \frac{1}{p_i p_j p_k} + \cdots \right]$$

$$\frac{n}{p_1 p_2 \cdots p_t}\left[p_1 p_2 \cdots p_t - \left(p_2 p_3 \cdots p_t + p_1 p_3 p_4 \cdots p_t - \cdots + \cdots p_1 \cdots p_{t-1}\right) + \left(p_1 \cdots p_{i-1} p_{i+1} \cdots p_{j-1} p_{j+1} \cdots p_t\right) + \cdots\right]$$

$(-1)^t$ ↑

$t=3$

$$\boxed{(p_1 - 1)(p_2 - 1)\cdots(p_t - 1)}$$

$$= p_1 p_2 \cdots p_t - p_2 p_3 \cdots p_t - p_1 p_3 \cdots p_t - \cdots$$

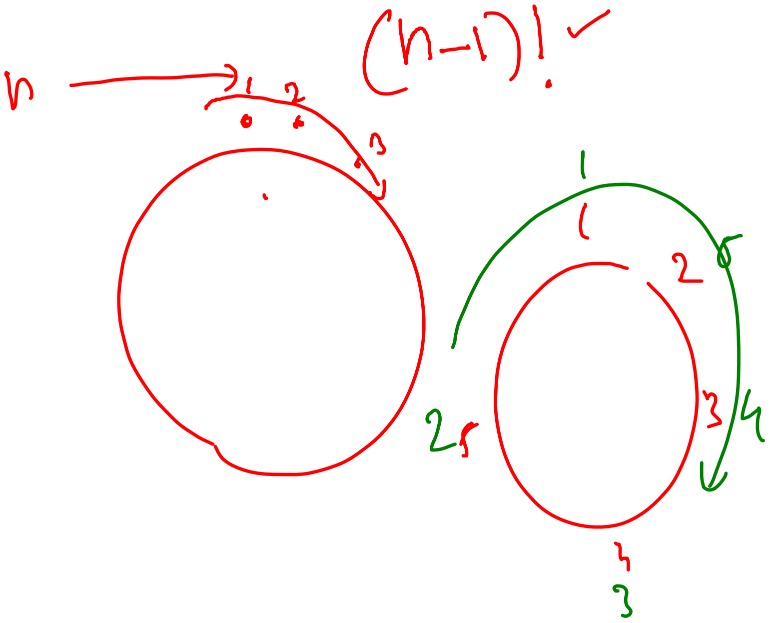
$$+ \cdots$$

$$\frac{n}{p_1 p_2 \cdots p_t}(p_1-1)(p_2-1)\cdots(p_t-1)$$

$$=\left[n \left(\frac{p_1-1}{p_1}\right)\left(\frac{(p_2-1)}{p_2}\right)\frac{(p_3-1)}{p_3}\cdots\frac{(p_t-1)}{p_t}\right.$$

$$=n\left[\left(1-\frac{1}{p_1}\right)\left(1-1/p_2\right)\cdots\left(1-1/p_t\right)\right]$$

$$=n\prod_{i=1}^{t}\left(1-1/p_i\right)$$



$\bigcup$ ↓

12 people

6 married couple

$(11!)$

P_1 → couples sits adjacent

A_1

H_1

P_2

$\boxed{h_1 \cancel{x} w_1}$ → = 11 $h_1 w_1$

⋮

$\binom{6}{1}$ $\underbrace{2.10!}$

P_6

$$11! - \binom{6}{1} 2 \cdot 10! + \binom{6}{2} 2^2\, 9!$$

$$- \binom{6}{3} \underbrace{2^3 \times 8!}$$

$$+ \binom{6}{4} 2^4\, 7! - \binom{6}{5} 2^5\, 6!$$

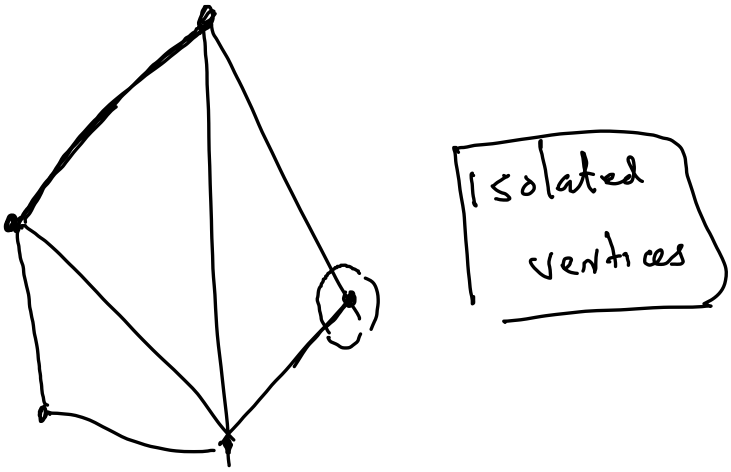
$$+ \binom{6}{6} 2^6\, 5!$$

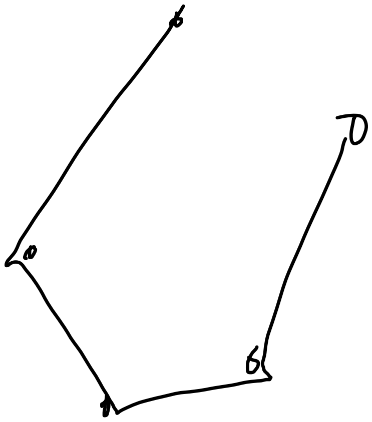
2 couples $\bigcirc\!\!\!\!H_1$ H_2

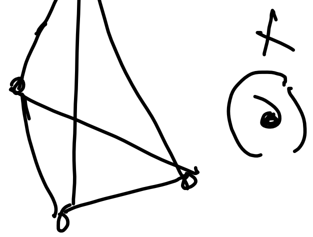
$\underset{\frown}{h_1 w_1} \to H_1$

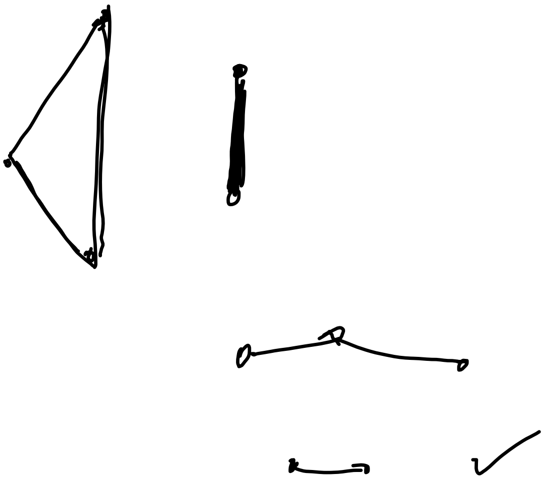
$h_2 w_2 \to H_2$

$(2^2 \cdot 9!)$ $\qquad 10 \to 9!$


Isolated vertices







How many graphs are there on 5 vertices?

$$\binom{5}{2}$$

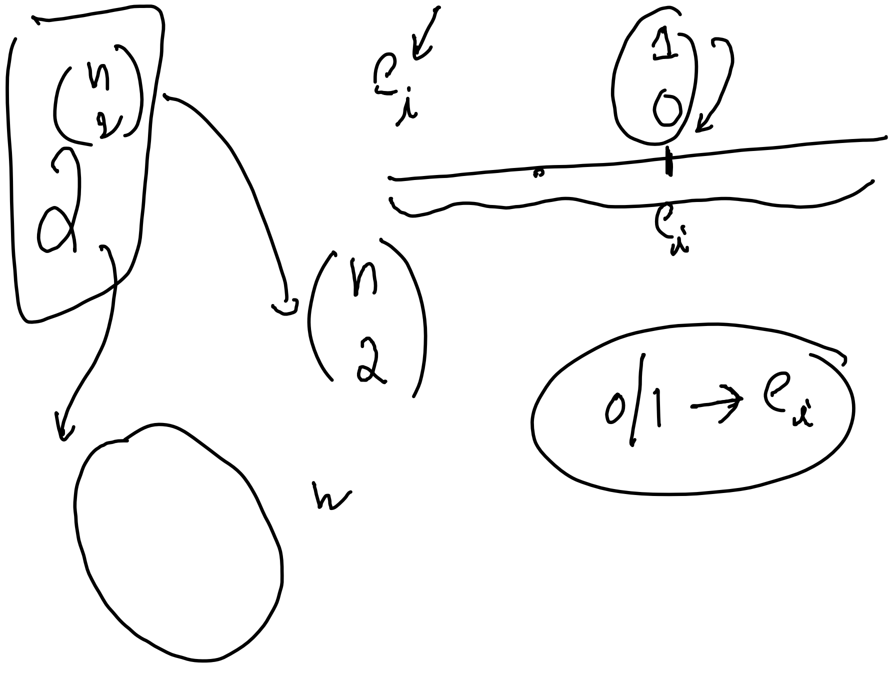
e_1 $(v_1\ v_2)$ $(v_2\ v_3)$ (v_3, v_4) $(v_4\ v_5)$

e_2 $(v_1\ v_3)$ (v_3, v_5)

e_4 $(v_1\ v_5)$ (v_2, v_5)

$2 + 1 = 10$

$4 + 3 + \binom{5}{2} = 10$



$n=5$

$\binom{n}{2}=\binom{5}{2}=10$

$2^{10}=|U|$

$A_1 \leftarrow P_1 \leftarrow v_1$ is isolated

$A_2 \leftarrow P_2 \leftarrow v_2$ "

$\vdots$

$A_5 \; \cdots P_5 \leftarrow v_5$ is isolated

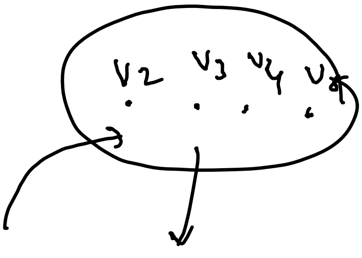
$$\underbrace{\underset{\checkmark}{\overline{A_1}} \cap \underset{\checkmark}{\overline{A_2}} \cap \cdots \cap \underset{\checkmark}{\overline{A_5}}}$$

$$= |U| - \sum_{i=1}^{5} |A_i| + \sum_{i,j} |A_i \cap A_j|$$

$$- \cdots (-1)^5 |A_1 \cap \cdots \cap A_5|$$

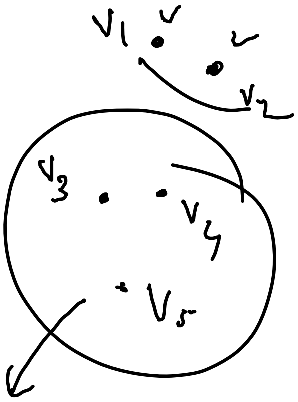
$$|A_{\underline{1}}| = 2^6$$

$$|A_i| = 2^6$$

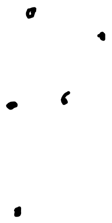


$$2^{\binom{4}{2}} = 2^6$$

$$|A_i \cap A_j| = 8$$



$2^{\binom{1}{2}} = 2^0 = 1$ $\quad 2^{\binom{2}{2}} = 2^1$

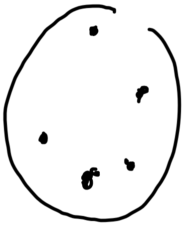


$2^{\binom{3}{2}} = 2^3 = 8$

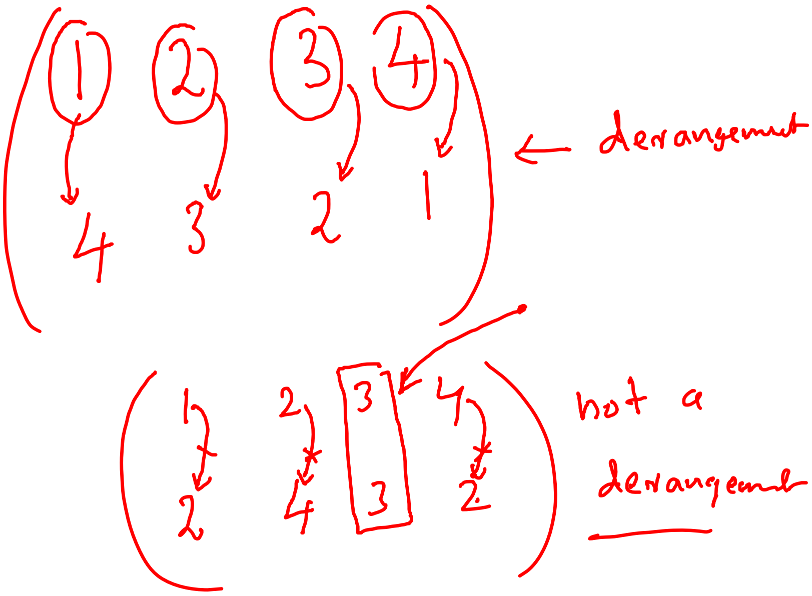
$$2^{10} - \binom{5}{1} 2^6 + \binom{5}{2} 2^3 - \binom{5}{3} 2^1$$

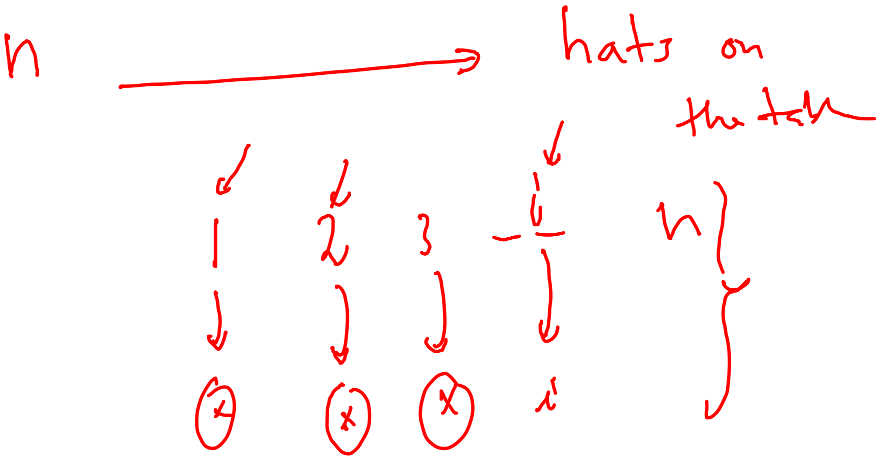
$$+ \binom{5}{4} 2^0 + \binom{5}{5} \cdot 1$$

$\binom{0}{2}$



$=1$

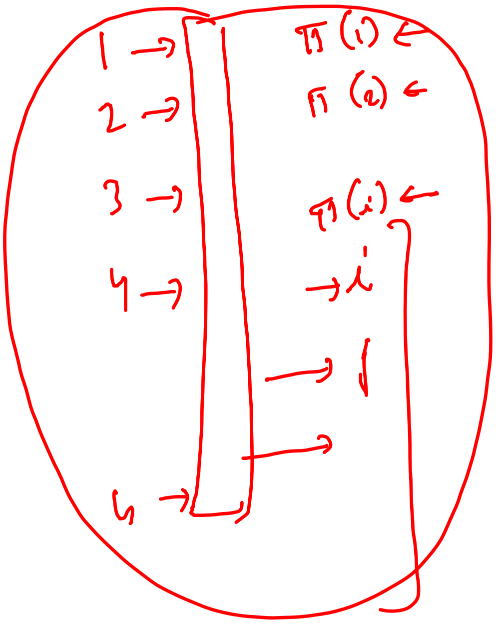

1 2 3 4
4 3 2 1
derangement
1 2 3 4
2 4 3 2
not a derangement


n
hats on the table
1 2 3 i
n
x x x i

n horses

1 . -- n

i ←



$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

U = set of all $n!$ permutations

$$|U| = n!$$

$P_1 \leftarrow$ at the 1st position

we see 1

$\pi(1)=1$

$P_i \leftarrow \pi(i)=i$

.

$P_n \leftarrow \pi(n)=n$

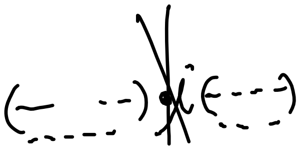
$P_1, P_2, \cdot \ \cdots \ P_n$

$A_i \subseteq U, \rightarrow$ does not satisfy P_i

$$\left| \underset{\smile}{\bar{A}_1} \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n \right| = ?$$

$$= |U| - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)!$$

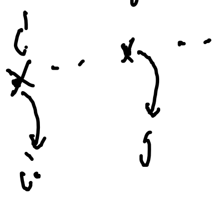
$$|A_1| = (n-1)!$$



$$|A_i| = (n-1)!$$

$$P_i \; ; \; P_j$$

$$|A_i \cap A_j| = (n-2)! \ldots\ldots$$



$$\boxed{n-3}$$

$$|\underset{\downarrow}{A_i} \cap \underset{\downarrow}{A_j} \cap \underset{\downarrow}{A_k}| = \underline{(n-3)!}$$

$$D_n = n! - \underbrace{\binom{n}{1}\overset{n!}{(n-1)!}} + \underbrace{\binom{n}{2}(n-2)!} \swarrow \frac{n!}{2!}$$

$$- \underbrace{\binom{n}{3}(n-3)!} + \cdots + \underline{(-1)^n (n-n)!}$$

$$= \overbrace{\frac{n(n-1)(n-2)}{3!}(n-3)!} = \frac{n!}{3!}.$$

$$(-1)^i \overbrace{\frac{n(n-1)\cdots n-(i+1)}{i!}(n-i)!}^{n!}$$

$$= (-1)^i \frac{n!}{i!}$$

$$D_n = n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots (-1)^n \frac{n!}{n!}$$

$$= n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots (-1)^n \frac{1}{n!} \right]$$ ✓

$$\approx n! \times e^{-1} = \boxed{n!/e}$$

$$e^{\overset{\swarrow -1}{x}} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\underbrace{e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots}$$

$$\boxed{D_n \approx \frac{n!}{e}} \rightarrow \boxed{\frac{1}{e}}$$

$$f(n) \rightarrow n$$

$$1 \overset{+1}{\,} 2 \overset{+1}{\,} 3 \overset{+1}{\,} 4 \overset{+1}{\,} 5 \overset{+1}{\,} 6$$

$$n = (n-1) + 1$$

$$\overset{10}{a_1},\ \overset{10+k}{a_2},\ \overset{10+2k}{a_3},\ \overset{10+3k}{a_4},\ \cdots\quad \overset{nth}{10+(n-1)k}$$

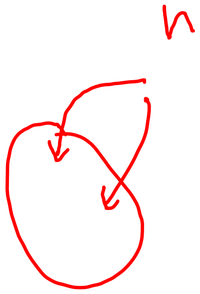
$$\boxed{a_n = a_{n-1} + k}$$

$$a_1,\ a_2,\ a_3,\ a_4\ \cdots$$

$$a_n = q\,a_{n-1}$$

$$a_2 = a_1 q \qquad a_3 = a_2 q$$
$$= a_1 q^2.$$

$$\cdots$$



Fibonacci sequences

$F_0 \leftarrow 0$

$\{F_1 \leftarrow 1$

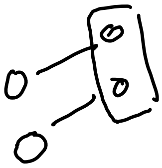
$F_2 \leftarrow 1$

$F_3 \leftarrow 1 + 1 = 2$

$$\begin{cases} F_4 = 2 + 1 = \textcircled{3} \\ F_5 = \underset{\overset{\uparrow}{3}}{F_4} + \underset{\overset{\uparrow}{2}}{F_3} = \textcircled{5} \end{cases}$$

# of existing pairs rabit

$$\underbrace{F_n} = \underbrace{F_{n-1}} + \underbrace{F_{n-2}}$$



$$\left[\begin{array}{c} \underline{F_n = F_{n-1} + F_{n-2}} \quad \underline{\text{for } n \geqslant 2} \\ \\ F_0 = 0, \quad F_1 = 1 \end{array}\right]$$

recurrence
relation

$$F_0 + F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - \underline{1}$$

$$\begin{array}{cccc} 0 & 1 & 1 & 2 \end{array}$$

$n = 0$

$n = 1$

$F_0 = 0$

$S_0 = F_0 = 0 \longrightarrow = F_2 - 1 = 1 - 1 = \underline{\underline{0}}$

$S_1 = F_0 + F_1 = 0 + 1 = 1 = F_3 - 1 = 2 - 1 = \underline{\underline{1}}$

induction on $\underline{\underline{n}}$

$$\underbrace{F_0 + F_1 + \cdots + F_n} + \underbrace{F_{n+1}} = F_{n+3} - 1$$

$$\downarrow$$

$$\underbrace{F_{n+2}} - 1 + \underbrace{F_{n+1}} \boxed{= \underbrace{F_{n+3}} - 1}$$

even	odd	odd	~~even~~	odd	odd	even	odd	odd	even
0,	1,	1,	2,	3,	5,	8,	- x -	x	x
F_0,	F_1,	F_2,	F_3	F_4	F_5	F_6			

F_n is even

F_{100} ?

$F_0 = 0 \quad F_1 = 1 \quad F_2 = 1 \quad F_3 = 2 \quad F_4 = 3$

$F_5 = 5 \quad F_6 = 8$ .....

$F_n \qquad n = 1000$?

$$\boxed{F_n} = \frac{1}{\sqrt{5}} \left(\frac{\sqrt{5}+1}{2} \right)^{n \leftarrow 0} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n}$$

$F_0 = 0$ //

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}-\frac{1-\sqrt{5}}{2}\right]$$

$$F_1 = \frac{1}{\sqrt{5}}\,\frac{2\sqrt{5}}{2} = 1$$

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2$$

$$\boxed{F_n - F_{n-1} - F_{n-2} = 0}$$ for $n \geq 2$

$$\boxed{F_n = q^n}$$ $q \neq 0$

$$q^n - q^{n-1} - q^{n-2} = 0$$ → for $n \geq 2$

devide both sides by q^{n-2} → q^{n-2}

$$\boxed{q^2 - q - 1 = 0}$$

$$x^2 - x - 1 = 0 \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$ax^2+bx+c=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$a=1$

$b=-1$

$c=-1$

$$=\frac{1\pm\sqrt{1+4}}{2}$$

$$\left\{\frac{1+\sqrt{5}}{2},\ \frac{1-\sqrt{5}}{2}\right\}$$

$$q=\frac{1+\sqrt{5}}{2} \text{ or } \frac{1-\sqrt{5}}{2}$$

$$\underline{q^n=q^{n-1}+q^{n-2}},\quad n\geq 2$$

$$\left.\begin{array}{l} F_0 = 0 \\ F_1 = 1 \end{array}\right\}$$ we have to make sure

$$F_0 = \left(\frac{\sqrt{5}-1}{2}\right)^{\overset{n=0}{0 \leftarrow}} = 1$$

$$\searrow F_0 = 0$$

$$F_n = q^n \leftarrow q = \left(\frac{\sqrt{5}+1}{2}\right)^n$$

$$\swarrow \qquad\qquad \swarrow \nearrow$$

$$q^n = q^{n-1} + q^{n-2} \quad n \geq 2$$

$$\left\{\begin{array}{l} F_0 = 1 \\ F_1 = \dfrac{\sqrt{5}+1}{2} \end{array}\right.$$

$$F_n = F_{n-1} + F_{n-2} \quad n \geq 2$$

$$F_n = q^n \quad \text{for } q = \frac{\sqrt{5}+1}{2}$$

$$q = \frac{1-\sqrt{5}}{2}$$

$$F_n = c \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n$$

$$F_n = c\left(\frac{\sqrt{5}+1}{2}\right)^{n-1} + c\left(\frac{\sqrt{5}+1}{2}\right)^{n-2}$$

$$= c\left[\underbrace{\left(\frac{\sqrt{5}+1}{2}\right)^{n-1} + \left(\frac{\sqrt{5}+1}{2}\right)^{n-2}}\right)$$

$$= C\left[\left(\frac{\sqrt{5}+1}{2}\right)^n\right]$$

$$F_n = c\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$F_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$F_{n-1} + F_{n-2} = F_n$$

$$c_1\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + c_2\left(\frac{1-\sqrt{5}}{2}\right)^{n-1} + c_1\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} + c_2\left(\frac{1-\sqrt{5}}{2}\right)^{n-2}$$

$$c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n = F_n$$

$$F_n = \underbrace{c_1}_{}\left(\frac{1+\sqrt{5}}{2}\right)^n + \underbrace{c_2}_{}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$F_0 = 0 = \overset{\swarrow}{c_1} + \overset{\swarrow}{c_2}$$

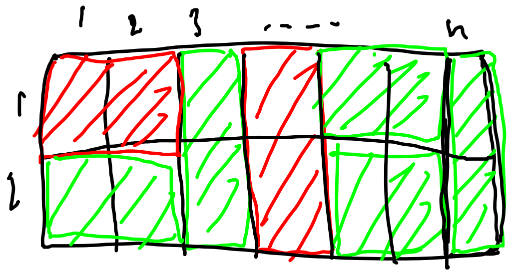
$$F_1 = 1 = \overset{\swarrow}{c_1}\left(\frac{1+\sqrt{5}}{2}\right) + \overset{\swarrow}{c_2}\left(\frac{1-\sqrt{5}}{2}\right)$$

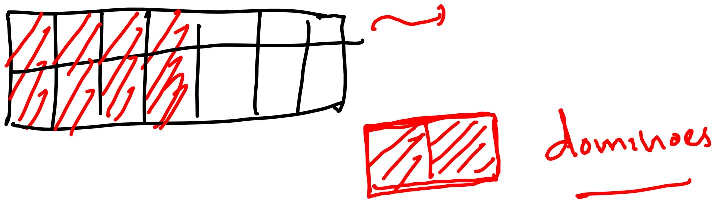
$$c_1 = \frac{1}{\sqrt{5}} \qquad c_2 = -1/\sqrt{5}$$

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\underset{\checkmark}{F_0} = 0 \qquad \underset{\checkmark}{F_1} = 1$$

$$F_n = F_{n-1} + F_{n-2}$$

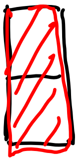


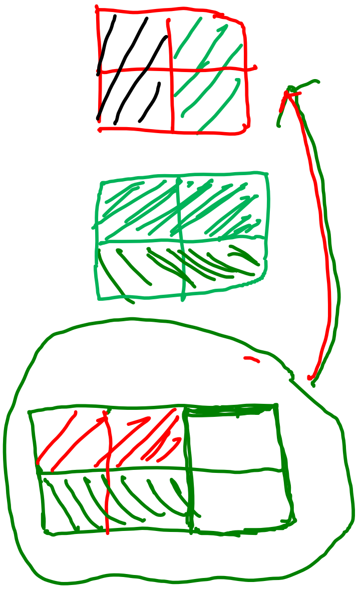


How many ways to perfectly cover $2 \times n$ brand ?

①

$a_0, a_1, a_2, \dots$





$0 \quad a_0 = 1, \ a_1 = 1, \ a_2 = 2$

0 1 1 2

$a_3 = 3$

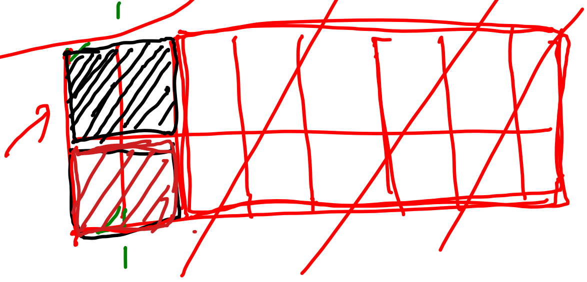
$2 + 1 = 3$

$\checkmark\ \checkmark\ \checkmark$

$a_0, a_1, a_2, a_3, \ldots a_{n-1} \ldots$

$$\boxed{a_n = a_{n-1} + a_{n-2}}$$

$n \geq 2$



$$\xcancel{\begin{aligned} a_n &= F_n \\ a_n &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \end{aligned}}$$

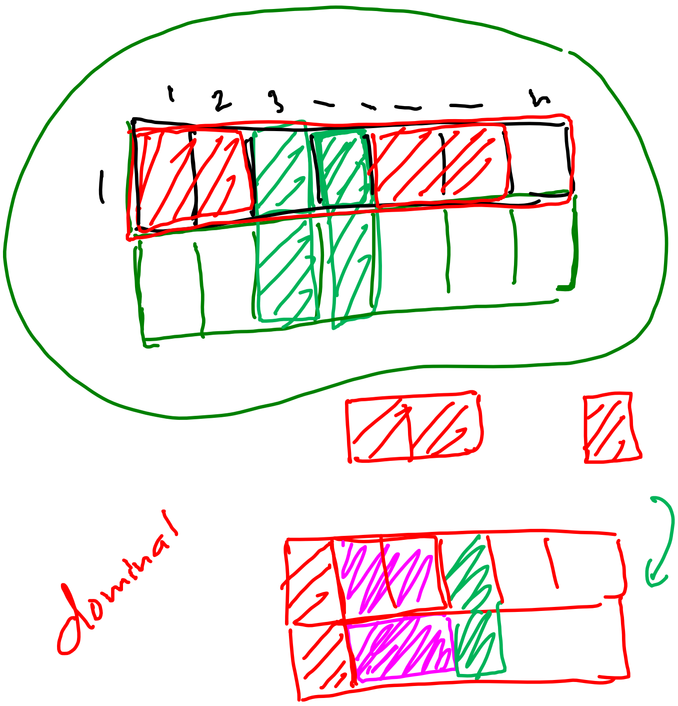
?

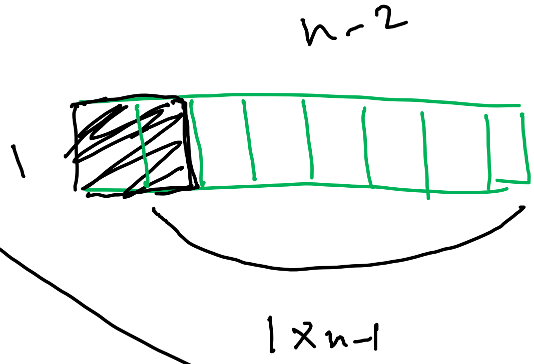
$$\begin{tikzcd}
a_0 \arrow[ddr] & a_1 \arrow[ddr] & a_2 \arrow[ddr] & a_3 & \cdots \\
1 & 1 & 2 & 3 & \\
F_0 & F_1 & F_2 & F_3 & \\
0 & 1 & 1 & 2 & 3
\end{tikzcd}$$

$$a_n = F_{n+1}$$

$$\left\{ a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \right.$$

$$\boxed{b_n \to a_n}$$





$$b_n = b_{n-1} + \underline{b_{n-2}} \leftarrow \boxed{n \geqslant 2}$$

$F_0 = 0$

$F_1 = 1$

$F_2 = 1$

$F_3 = 2$

$F_4 = 3$

$a_0 = 1$

$a_1 = 2$

$a_2 = 3$

$a_3 = 5$

$\boxed{a_n = F_{n+2}}$

?

$\emptyset = S_0$

$S_1 = \{1\}$

$S_2 = \{1, 2\}$

✱ 3

$S_3 = \{1, 2, 3\}$

$1 + 3 + 1 +$

$$S_n = \{1, 2, 3, \dots\dots n\}$$

up to n-1, the sequence

$a_0, a_1, - - - - , a_{n-1}$ $\boxed{a_n}$

$S_n = \{1, - - - , n\}$

$\mathcal{F} = \{A \subseteq S_n :$ A does not have consecutive integers in it$\}$

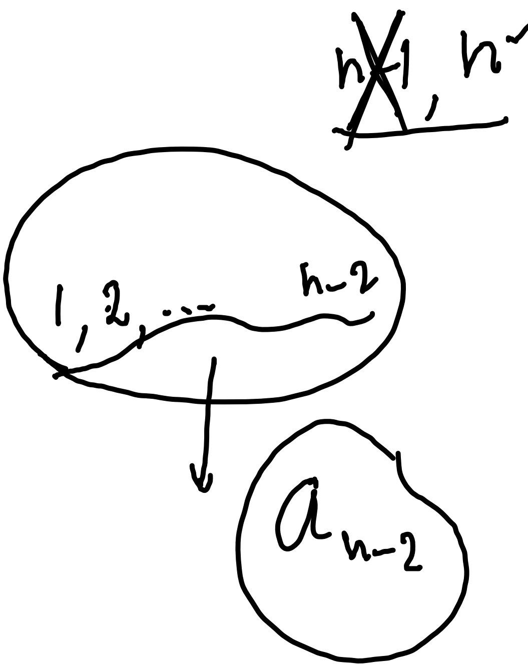
↙

$\mathcal{F}' \subseteq \mathcal{F}$

$\mathcal{F}'' \subseteq \mathcal{F}$

$\mathcal{F}' = \{A \in \mathcal{F} : n \in A\}$

$\mathcal{F}'' = \{A \in \mathcal{F} : n \notin A\}$



for each $A \in F'$

$A' = A \setminus \{n\}$

F''

$\{1, 2, \dots n-1\}$

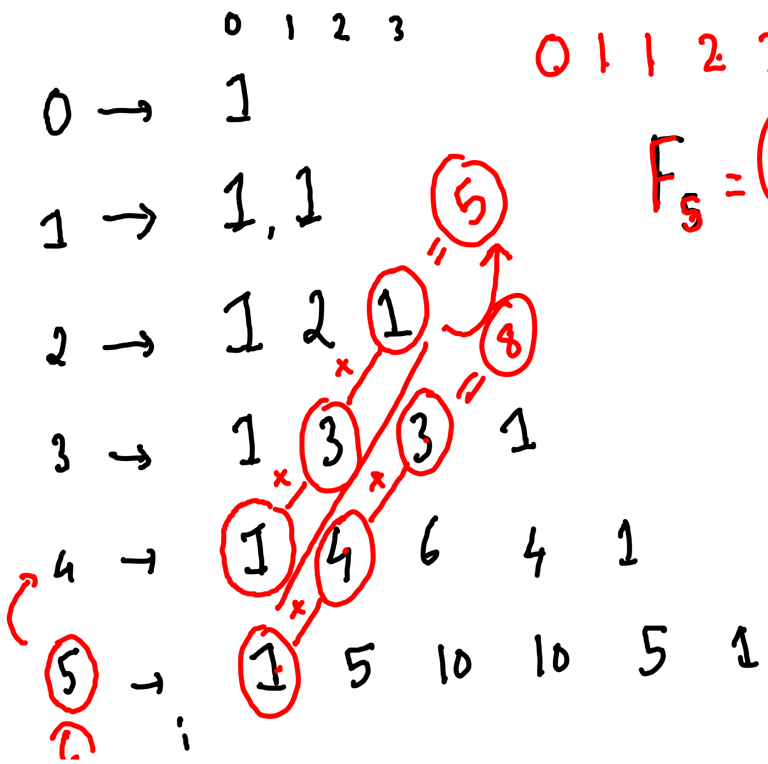
$|F''| = a_{n-1}$

$$|F| = |\overset{\downarrow}{F'}| + |F''|$$

$$\left\{ \begin{array}{l} \boxed{a_n = a_{n-2} + a_{n-1}} \checkmark \\ \qquad\qquad n \geq 2 \end{array} \right.$$

$$a_n = F_{n+2}$$

$$= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+2} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+2}$$



0 1 1 2 3 ⑤ 8

$$F_5 = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-2}{2}$$

$$\downarrow \qquad \downarrow$$

$$\binom{4}{0} + \binom{3}{1} + \binom{2}{2}$$

$$h_0, h_1, h_2, h_3, h_4, h_5, \overbrace{\cdots}^{\checkmark \times \times} \boxed{h_n} \cdots$$

$\boxed{\text{for } n \geq k}$

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k} + b_n$$

Order is k, constant coefficients

$$\boxed{F_n} = F_{n-1} + F_{n-2} + \boxed{b_n}$$

$a_1 = 1$

$a_2 = 1$

$k = 2$

$\underset{=}{0}$

Homogenous

$$h_n = \underbrace{q}\, h_{n-1}$$

$\underbrace{a_1 = q}$ $\underline{b_n = 0}$

$$\underline{h_n = n\,h_{n-1}}$$

$$\boxed{h_n = n!}$$

$$\boxed{a_1 = n}$$

$$\underline{b_n = 0}$$

$$\left[\begin{array}{l} h_0 = 1 \\ h_1 = 1 \\ h_2 = 2.1 = 2 \\ h_3 = 3\text{-}2\text{-}1 = \\ h_4 = 4.h_3 \\ \quad = 4! \end{array}\right.$$

$$\boxed{h_{k-1} = a_1 h_{k-2} + \cdots}$$

$$+ a_k h_{k-1}$$

$$\underline{\qquad\qquad}\;k$$

$$\boxed{\cancel{a_k h_{-1}}}$$

k

$$\underbrace{h_0, h_1, \dots\Big]\, h_{k-1}}$$

$h_0, h_1, h_2, h_3, \text{-----} \underbrace{h_n}$

$$\boxed{h_n} = c_1 h_{n-1} + c_2 h_{n-2} + \cdots + \boxed{c_k} \boxed{h_{n-k}}$$

$\cancel{\text{"}\underbrace{h_i h_j}\text{"}}$ "k" $C_k \neq 0$

$$\boxed{h_n = \overset{n^2}{c_1} h_{n-1} + \cdots + c_k h_{n-k}} + \underbrace{b_n}$$

$\boxed{n \geq k}$ $\quad \boxed{c_i} \leftarrow$ $\quad \boxed{b_n = 0}$

$h_n = \boxed{n} h_{n-1}$

$k - k = 0 \quad h_0$

$\boxed{\begin{matrix} h_1 = 1 \\ h_0 = 1 \end{matrix}}$

$$h_{k-1} = \cdots + c_k h_{\underbrace{k-1-k}_{\boxed{h_{-1}}}}$$

$$h_0, h_1, h_2, \cdots$$

Characteristic equation of
the recurrence relation

$$h_n = 2h_{n-1} + h_{n-2}$$

$$\downarrow \quad \boxed{q^n} \rightarrow \quad q^n =$$

$$\boxed{h_n = q^n}$$

$$h_0 = q^0, \quad h_1 = q^1, \quad h_2 = q^2 \ldots$$

$$\boxed{h_n = 2h_{n-1} + h_{n-2}} \quad \boxed{n \geq 2}$$

$$\boxed{q \neq 0}$$

$$\overset{q^2}{\cancel{q^n}} = \overset{2q}{\cancel{2q^{n-1}}} + \cancel{q^{n-2}} \quad ✓$$

$$q^2 = 2q + 1 \quad ✓$$

$$\boxed{x^2 - 2x - 1 = 0}$$

✓

$$q^n = 2q^{n-1} + q^{n-2}$$

$$\boxed{h_0 = a \quad h_1 = b}$$

$$\boxed{q^0 = h_0 \quad q^1 = h = b}$$

$q = \underline{1+\sqrt{2}\}}$ $\quad q^1 \quad ax^2+bx+c=0$

$r = 1-\sqrt{2}$

$2+2\sqrt{2} \quad 2-2\sqrt{2} \quad \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$\dfrac{\boxed{2+\sqrt{8}}}{2} \;/\; \dfrac{2-\sqrt{8}}{2} \longleftarrow \dfrac{2 \pm \sqrt{4+4}}{2\times 1}$$

$q^0 = h_0 = \underline{10}$ ✓

$q^1 = h_1 = \underline{5}$ ✓

$(1+\sqrt{2})^0 = 10$ ✓ ✗

$(1-\sqrt{2})^0 = 10$ ✓

$$\underbrace{q_1 = 1+\sqrt{2}} \qquad \underbrace{q_2 = 1-\sqrt{2}}$$

$$\boxed{h_n} = 2h_{n-1} + h_{n-2}$$

$$h_n = \underbrace{c_1 q_1^n + c_2 q_2^n}$$

$$[c_1 q_1^n + c_2 q_2^n] =$$

$$2[c_1 q_1^{n-1} + c_2 q_2^{n-1}] + [c_1 q_1^{n-2} + c_2 q_2^{n-2}]$$

$$c_1 q_1^n = 2c_1 q_1^{n-1} + c_1 q_1^{n-2} \quad \text{—①}$$

h_n

$$\underbrace{c_2 q_2^n} = 2c_2 q_1^{n-2} + c_2 q_2^{n-2} \quad \text{—⑪}$$

$$[c_1 q_1^n + c_2 q_2^n] = 2\overbrace{[c_1 q_1^{n-1} + c_2 q_1^{n-2}]}^{h_{n-1}} + \underbrace{c_1 q_1^{n-2} + c_2 q_2^{n-2}}_{h_{n-2}}$$

$$h_n \to \underbrace{c_1}_{}(1+\sqrt{2})^n + \underbrace{c_2}_{}(1-\sqrt{2})^n$$

$$\boxed{h_n = 2h_{n-1} + h_{n-2}} \longrightarrow \left\{ \begin{array}{l} \underbrace{c_1}_{} + \underbrace{c_2}_{} = 10 \\ \underbrace{c_1}_{}(1+\sqrt{2})^1 + \underbrace{c_2}_{}(1-\sqrt{2})^1 = 5 \end{array} \right.$$

$$h_0 \leftarrow 10 \qquad h_1 = 5 \longrightarrow$$

$$q_1^n \qquad q_2^n$$

$$h_n = \underbrace{c_1 q_1^n + c_2 q_n^n}_{}$$

$k = 2$

$$h_n - a_{\underset{1}{}} h_{n-1} - \cdots - a_k h_{n-k} = 0$$

$$\boxed{\begin{array}{l}(a_k \neq 0)\\ \text{and } n \geq k\end{array}}$$

$$h_n = q^n \qquad q \neq 0$$

$$q^n - a_1 q^{n-1} - a_2 q^{n-2} - \cdots - a_k q^{n-k} = 0$$

$$\left[q^k - a_1 q^{k-1} - \cdots - a_k = 0\right]$$

$$\boxed{x^k - a_1 x^{k-1} - \cdots - a_k = 0}$$

~~non-zero~~ distinct roots ✓

$$c_1, \quad c_2, \ldots, c_k$$

$$h_n = \underbrace{\left[c_1 q_1^n + c_2 q_2^n + \cdots + c_k q_k^n\right]}$$

$$h_n = q_i^n \qquad q_i$$

$$\begin{cases} h_0 = - \\ h_1 = - \\ h_2 = - \\ \vdots \\ h_{k-1} = - \end{cases}$$

$\underline{n \geq k}$

$$\underline{x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_k} = 0$$

$q_1, q_2, \cdots, q_k \leftarrow$

$$h_n = c_1 q_1^n + c_2 q_2^n + \cdots + c_k q_k^n$$

$$\begin{array}{ll}
① - & \overset{\checkmark}{c_1}1 + \overset{\checkmark}{c_2}1 + \cdots + \overset{\checkmark}{c_k}1 = h_0 \\
① - & c_1 q_1 + c_2 q_2 + \cdots + c_k q_k = h_1 \\
\vdots & c_1 q_1^2 + c_2 q_2^2 + \cdots + c_k q_k^2 = h_2 \\
 & \vdots \\
⑫ \leftarrow & c_1 q_1^{k-1} + c_2 q_2^{k-1} + \cdots + c_k q_k^{k-1} = h_{k-1}
\end{array}$$

Vandermonde matrix

$$\overbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \cdots 1 \\ q_1^1 & q_2^1 & \cdots & q_k^1 \\ q_1^2 & q_2^2 & \cdots & q_k^2 \\ q_1^{k-1} & q_2^{k-1} & \cdots & q_k^{k-1} \end{bmatrix}}^{k} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{k-1} \end{bmatrix}$$

$$\boxed{\prod_{1\le i<j\le k}(q_j - q_i) \neq 0} \checkmark \quad \boxed{Ax=b}$$

$$n \geq 3 \leftarrow \left[h_n = \underbrace{2h_{n-1} + h_{n-2} - 2h_{n-3}} \right]$$

$$h_0 = 1 \quad h_1 = 2 \quad h_2 = 2$$

$$\left[x^3 - 2x^2 - x + 2 = 0 \right]$$

$$(x = 1, -1, 2)$$

$$h_n = 1^n \longrightarrow 1 \quad 1\,1 \quad 1\,1\,1\,1$$

$$c_1 1^n + c_2(-1)^n + c_3 2^n$$

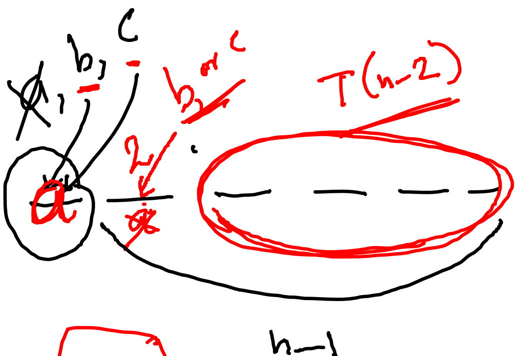
$$\left[\begin{array}{l} c_1 - c_2 + c_3 = 1 \leftarrow h_0 = 1 \\ c_1 1 - c_2 \end{array} \right]$$

a, b, c

aabc

aba~~a~~c

T(n)



$\boxed{T(0) = 1}$

$T(1) = 3$

$2 \times 3 +$

2×1

$= 8$

$T(n) = 2\,T(n-1)$

$+ 2\,T(n-2)$

$\boxed{2 \cdot T(n-1)}$

$+$

$2\,T(n-2)$

$$T(n) = 2\,T(n-1) + 2\,T(n-2)$$ ✓

$$\boxed{\begin{array}{l} T(0)=1 \\ T(1)=3 \end{array}}$$

$$x^2 - 2x - 2 = 0$$

$$\begin{array}{l} 1+\sqrt{3} \\ 1-\sqrt{3} \end{array} \leftarrow \frac{2 \pm 2\sqrt{3}}{2} = \frac{2 \pm \sqrt{4+8}}{2}$$

$$T(n) = c_1\left(1+\sqrt{3}\right)^n + c_2\left(1-\sqrt{3}\right)^n$$

$$\boxed{n \geq 2}$$

$T(0)=1$ ✓ and $T(1)=3$ ✓✓

$$c_1 + c_2 = 1$$

$$c_1\left(1+\sqrt{3}\right) + c_2\left(1-\sqrt{3}\right) = 3$$ ✓

$$c_1 = \frac{2+\sqrt{3}}{2\sqrt{3}}, \quad c_2 = \frac{-2+\sqrt{3}}{2\sqrt{3}}$$

Brualdi ✓ ← Grimaldi and Ramana

$$h_n = 4h_{n-1} - 4h_{n-2} \quad (n \geq 2)$$

$$x^2 - 4x + 4 = 0 \quad \text{— ①}$$

$$(x-2)^2 = 0 \Rightarrow x = 2$$

$\boxed{h_0 = 1}$

$h_1 = 3$

$\underline{c_1 2^n + c_2 2^n}$

$(c_1 + c_2) 2^n = \underline{c' 2^n}$

$\boxed{\underline{c' = 1}}$

$c' \cdot 2 = \underline{1 \cdot 2 = 2 \neq 3}$

$\boxed{2^n} , \boxed{n 2^n}$

$\underline{h_n = 4 h_{n-1} - 4 h_{n-2}}$

$n 2^n = 4 \underbrace{(n-1) 2^{n-1}} - 4 \underbrace{(n-2) 2^{n-2}}$

$n = 2(n-1) - (n-2)$

$\boxed{n = n}$

$$\boxed{h_n = a_1 h_{n-1} + --- + a_k h_{n-k}}$$

$n q^n \leftarrow$

$q^n, \; n q^n$

$q^n, \; n q^n, \; n^2 q^n$

$$\underline{q^n, \; n q^n, \; n^2 q^n, \; ---, \; n^{s-1} q^n}$$

$\boxed{c n^i q^n}$ $\quad$ $\boxed{0 \leq i \leq s-1}$

$$\underline{H_i^{(n)} = c_1 q_i^n + c_2 n q_i^n + c_3 n^2 q_i^n + --- + c_s n^{s-1} q_i^n}$$

$$h_n = \left(C_1^1 q_1^n + C_2^1 n q_1^n + C_3^1 n^2 q_1^n + \cdots + C_{s_1}^1 n^{s_1-1} q_1^n\right)$$

$$+ \left(C_1^2 q_2^n + \cdots + C_{s_2}^2 n^{s_2-1} q_2^n\right)$$

$$- - - - -$$

$$+ \left(C_1^t q_t^n + \cdots + C_{s_t}^t n^{s_t-1} q_t^n\right)$$

$$h_n = -h_{n-1} + 3h_{n-2} + 5h_{n-3} + 2h_{n-4}$$

$n \geq 4$

$h_0 = 1$

$h_1 = 0$

$h_2 = 1$

$h_3 = 2$

$$\boxed{x^4 + x^3 - 3x^2 - 5x - 2 = 0}$$

$$\boxed{x = -1, -1, -1, 2}$$

$$\begin{tikzcd}
q_1=-1 \arrow[ddr, bend right] & s_1=3, & H_n^{(2)}=d2^n \arrow[ddl, leftrightarrow, "+"] \\
q_2=2 & s_2=1 \arrow[ur] & \\
 & H_n^{(1)}=c_1(-1)^n+c_2n(-1)^n+c_3n^2(-1)^n &
\end{tikzcd}$$

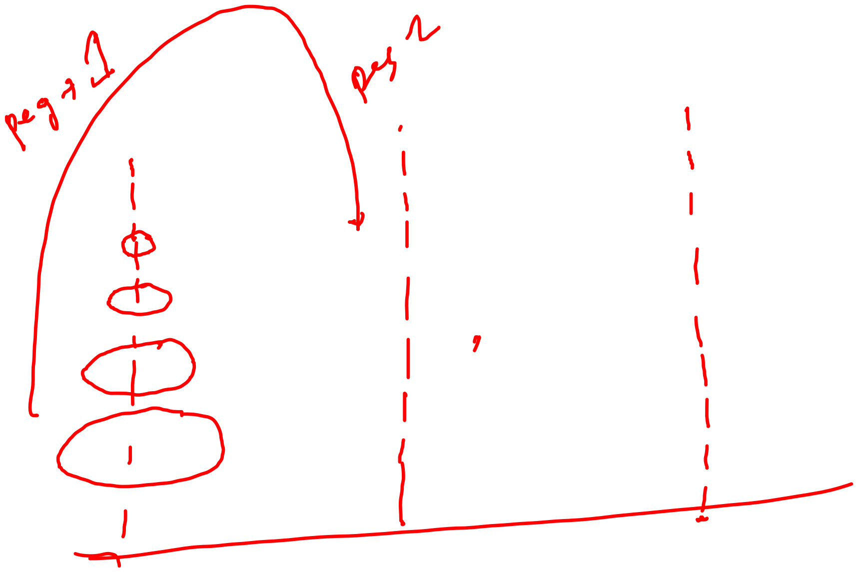
$$\begin{aligned}
h_n &= H_n^{(1)} + H_n^{(2)} \\
&= c_1(-1)^n + c_2 n(-1)^n + c_3 n^2(-1)^n \\
&\quad + d\,2^n
\end{aligned}$$

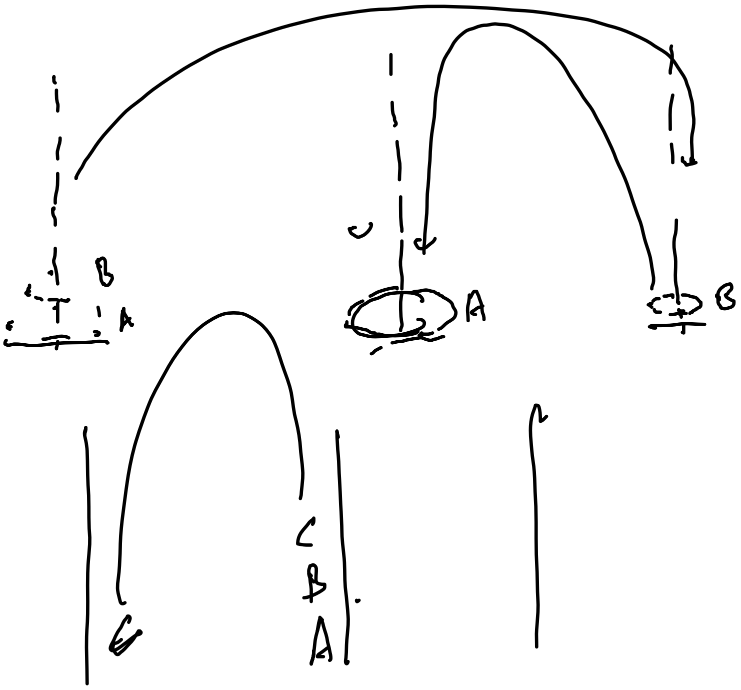
$$\begin{cases} n=0 & c_1+d=1 \\ n=1 & \end{cases}$$

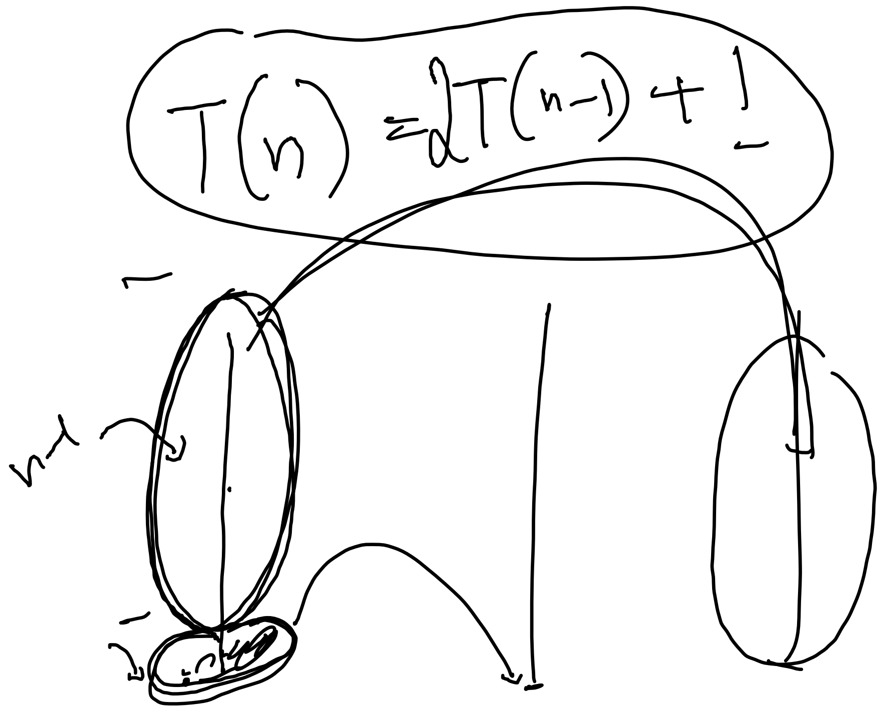
$$h_n = c_1 h_{n-1} + \cdots + c_k h_{n-k} + b_n$$

$$b_n \neq 0$$

Tower of Hanoi problem







$$T(n) = 2\,T(n-1) + 1$$

$$= 2\left(2\,T(n-2) + 1\right) + 1$$

$$= 4\,T(n-2) + 2 + 1$$

$$= 2^{n-1}\left(T(0) + 1\right) + \cdots + 2 + ?$$

$$\underline{= 2^{n-1} + \cdots + 2}$$

$$\underline{\underline{= 2^{n} - 1}}$$

$$\underbrace{h_n = \underset{\uparrow\, a_1}{2}\, h_{n-1}} + \overset{b_n}{\boxed{\underline{\underline{1}}}}$$

$$\underbrace{h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k}} + \underline{\underline{\boxed{b_n}}}$$

$$\underbrace{h_n = 2h_{n-1}}\ \cancel{+b_n} \leftarrow 1$$

$$h_n - 2h_{n-1} = 0$$

$$x - 2 = 0 \Rightarrow \underline{\underline{x=2}}$$

$$\boxed{h_n = c \cdot \underline{2}^n} \quad \boxed{\begin{array}{l} h_0 = 0 \\ h_1 = 1 \end{array}}$$

$$h_n = 2h_{n-1} + 1$$

$h_n = k$

$k, k, k, k, k, \ldots$

$\Rightarrow$ $\underline{\qquad\qquad}$

0 $\cancel{k} = \cancel{2k} + 1$ $\quad k$

$h_n = 2h_{n-1} + 1$

$k = -1 \Rightarrow$

$2 \times -1 + 1 = -1$

$h_1 = 1$

✓

$-1, -1, -1, -1, -1, \cdots$

$h_n = c\,2^n \rightarrow h_n = 2h_{n-1}$

$h_n = -1 \rightarrow h_n = 2h_{n-1} + 1$

$h_1 = 1$

$\boxed{h_n = c\,2^n - 1} \rightarrow \boxed{h_n - 2h_{n-1} - 1 = 0}$

$$c2^n-1 = c2-1 = h_1 = 1$$

$n=1$

$$c2 = 2$$

$$c=1$$

$$h_n = c2^n-1 = \boxed{2^n-1}$$

$$h_n = 3h_{n-1} - 4n$$

$$h_0 = 2$$

① $h_n = 3h_{n-1}$

$$x-3=0 \Rightarrow x=3$$

$$h_n = c\cdot 3^n$$

$$h_n = 3h_{n-1} - 4n$$

$$\boxed{h_n = \underline{\underline{r}}\,n + \underline{\underline{s}}}$$

$$rn + s = 3\left(r(n-1) + s\right) - 4n$$

$$= 3rn - 3r + 3s - 4n$$

$$rn + s = \underbrace{(3r-4)}\,n + \underbrace{(3s-3r)}$$

$$r = 3r - 4 \Longrightarrow 2r = 4 \Rightarrow \boxed{r = 2}$$

$$s = 3s - 3r$$

$$2s = 3r = 6 \Rightarrow \boxed{s = 3}$$

$\boxed{2n+3} \leftarrow \boxed{h_0=3}$ $\boxed{h_0=2}$

$$\boxed{h_n = 3h_{n-1} - 4n}$$

$$2n+3 = 3\left(2(n-1)+3\right) - 4n$$

$$= 6\cancel{n} + 3 - \cancel{4n} = 2n+3$$

$$\boxed{h_n = \underline{2n+3} - 3^n}$$

$$\boxed{h_n = 3h_{n-1} - 4n}$$

$$h_0 = 2$$

$$3 + c = 2$$

$$\boxed{c = -1}$$

$$\underbrace{h_n = 2h_{n-1} + 3^n}, \quad \boxed{h_0 = 2} \nwarrow$$

① → $h_n = 2h_{n-1}$,

$$x - 2 = 0 \Rightarrow \boxed{x = 2}$$

$$h_n = c2^n$$

$$h_n = 2h_{n-1} + 3^n$$

$$h_n = p3^n \Rightarrow p\overset{3}{\cancel{3}}^{n} = 2p\cancel{3}^{n-1} + \overset{3}{\cancel{3}}^{n}$$

$$p3\cancel{p} = \cancel{2p} + 3$$

$n=0$

$3^1=3$

3^{n+1}

$$p = 3$$

$$h_n = 3\times 3^n \Rightarrow h_n = 3^{n+1}$$

$$\boxed{h_n = 2h_{n-1}^{\times 3^n} + 3^n}$$

$$h_n = c \cdot 2^n + 3^{n+1}$$

$$2 = h_0 = c + 3$$

$$\boxed{c = -1}$$

$$\boxed{h_n = -2^n + 3^{n+1}}$$

$$\boxed{h_0 = 2}$$

$$\underbrace{h_n = 3h_{n-1}}_{} + 3^n, \quad \boxed{h_0 = 2}$$

$$\left.\begin{array}{l} h_n = 3h_{n-1} \\ \\ x - 3 = 0 \Rightarrow \underline{x = 3} \end{array}\right\} \quad \underline{h_n = c \cdot 3^n}$$

$$\boxed{h_n = 3h_{n-1} + 3^n}$$

$$\boxed{h_n = p3^n} \swarrow$$

$$\cancel{p3^n} = 3\left(p3^{n-1}\right) + 3^n$$

$$= \cancel{p3^n} + 3^n$$

$$\boxed{\underline{p = p + 1}}$$ ✗

$$\boxed{h_n = p n 3^n}$$

$$h_n = 3 h_{n-1} + 3^n$$

$$p n \cancel{3^n} = \cancel{3} p (n-1) \cancel{3^{n-1}} + \cancel{3^n}$$

$$\cancel{pn} = \cancel{pn} - p + 1$$

$$\boxed{p = 1} \leftarrow$$

$h_n = p_n 3^n = \boxed{\underline{n 3^n}} = \underline{6}$

$n=0$

$n 3^n = 3(n-1)3^{n-1} + 3^n$

$\boxed{n 3^n = n 3^n} \cancel{- 3^n + 3^n}$ ✓

$c 3^n + n 3^n$

when $n=0$, $\boxed{c = 2}$

$2 3^n + n 3^n$

$\boxed{= (n+2) 3^n}$

$$\begin{tikzcd}
h_0, h_1, h_2, h_3, \cdots \arrow[d, shift left] \\
g(x) = h_0 + h_1 x^1 + h_2 x^2 + h_3 x^3 + \cdots \arrow[u, shift left]
\end{tikzcd}$$

$$\begin{tikzcd}
h_0 \arrow[d] & h_1 \arrow[d] & h_2 & 0 & 0 & 0 & 0 \cdots \\
\binom{m}{0}, & \binom{m}{1}, & \binom{m}{2}, & \binom{m}{3}, \cdots
\end{tikzcd}$$

$$h_m = \binom{m}{m},$$

$$\underset{\overset{\uparrow}{h_0}}{\binom{m}{0}} + \underset{h_1}{\binom{m}{1}} x^1 + \underset{h_2}{\binom{m}{2}} x^2 + \binom{m}{3} x^3 + \cdots$$

$$\cdots + \underset{\overset{\uparrow}{h_m}}{\binom{m}{m}} x^m$$

$$\begin{tikzcd} g(x) = (1+x)^m & \\ & \binom{m}{0}, \binom{m}{1}, ---, \binom{m}{m} \arrow[ul, bend right] \end{tikzcd}$$

$$\begin{array}{cccccc} h_0 & h_1 & h_2 & h_3 & \cdots & \\ 1, & 1, & 1, & 1, & \cdots & \text{---} \end{array}$$

$$\downarrow$$

$$1 + x + x^2 + x^3 + \cdots\cdots$$

$$= \frac{1}{1-x} = g(x)$$

$$\begin{array}{ccccccccccc} 1, & 1, & 1, & 1, & \cdots, & 1, & 0 & 0 & 0 & 0 \\ h_0 & h_1 & h_2 & & & h_n & & & & \end{array}$$

$$\downarrow$$

$$1 + x + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$$

$$(1-x)(1+x+\cdots+x^n) = \underline{\underline{1-x^{n+1}}}$$

$$(1 + \cancel{\cdots + x^n}) - (\cancel{x + x^2 + \cdots + x^n} + x^{n+1})$$

$$(1-x)\underbrace{(1+x+x^2+\cdots\cdots)}=1$$

$$1+x+\cdots+\cdots=\frac{1}{1-x}$$

$$|x|<1$$

$$\frac{1}{1-x}=\underbrace{1+x+x^2+\cdots\cdots}$$

$$g(x)=\frac{1}{1-x}$$

1, 1, 1, 1, 1, ...

n=0 n=1

1, 2, 3, ...

$$\frac{x}{(1-x)^2}=0+\underbrace{x+2x^2+3x^3+\cdots}$$

$h_n = n+1$

$\boxed{\dfrac{x}{(1-x)^2}}$ 1

0, 1, 2, 3, ---

$\bigcirc\; h_n = n$

$0, 1, 2^2, 3^2, 4^2, 5^2, \ldots$

$$g(x) \leftarrow \dfrac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \cdots$$

$$x \dfrac{d}{dx}\left(g(x)\right) = 1 + \underbrace{2^2}_{} x + \underbrace{3^2}_{} x^2 + \cdots$$

$$= x + 2^2 x^2 + 3^2 x^3 + \cdots$$

$h_0, h_1, h_2, h_3, \ldots$

$$g(x) = \underline{h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \cdots}$$

$$\underline{\cdots + h_n x^n + \cdots}$$

$h_n = 1$

$1, 1, 1, 1, \ldots$

$$g(x) = \frac{1}{1-x} = 1 + x + x^2 + \cdots$$

$1=2^0,\ 2^1,\ 2^2,\ 2^3, \ldots$

$$\boxed{\frac{1}{1-x}} = 1+x+x^2+x^3+\cdots$$

$y=2x$,

$$\boxed{\frac{1}{\underline{1-2x}}} = \boxed{\frac{1}{1-y}} = 1+y+y^2+y^3+\cdots$$

$$= 1+2x+(2x)^2+(2x)^3+\cdots$$

$$= \underline{1+2x+2^2x^2+2^3x^3+}$$

$$\underline{\cdots + \underset{\smile}{2}^n x^n + \cdots}$$

$$\underline{1,\ 2,\ 2^2,\ 2^3,\ ---\ \boxed{2^n,}\ ---}$$

$$\downarrow$$

$$1 + 2x + 2^2x^2 + \underbrace{2}^{3}\,x^3 + ---$$

$$\frac{1}{1-2x} = (1-2x)^{-1}$$

$$(1+x)^{\boxed{\alpha}} = \underline{\binom{\alpha}{0} + \binom{\alpha}{1}(x) +}$$

$$\underline{\binom{\alpha}{2}(x)^2 + \binom{\alpha}{3}(x)^3 + ---}$$

$$\binom{\alpha}{0}, \binom{\alpha}{1}, \binom{\alpha}{2}, \cdots \binom{\alpha}{n}$$

$$\swarrow \qquad \swarrow \qquad \cdots$$

$$(1+x)^{\alpha}$$

$$(1+x)^{n} \leftarrow \textcircled{α}$$

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \cdots, \binom{n}{n}$$

$$, 0, 0, 0, 0$$

$$\frac{1}{(1+x)} = (1+x)^{-1}$$

$$= \binom{-1}{0} + \binom{-1}{1} x$$

$$+ \binom{-1}{2} x^{2} + --- + \binom{-1}{n} x^{n} + ---$$

$$\binom{-1}{n} = \frac{(-1)(-2)(-3)\cdots(-n)}{n!} = \swarrow (-1)^{n}$$

$$= (-1)^{n} \frac{\cancel{n!}}{\cancel{n!}} = (-1)^{n}$$

$$(1\underbrace{-x}_{y})^{-1} = \binom{-1}{0} + \binom{-1}{1}(-x) + \binom{-1}{2}(-x)^2 + \binom{-1}{3}(-x)^3 + \cdots$$

$$\cdots + \binom{-1}{n}(-x)^n + \cdots$$

$$\binom{-1}{n}(-x)^n = \cancel{\boxed{\binom{-1}{n}}(-1)^n}\, x^n$$

$$x^n$$

$$(-1)^n(-1)^n = (-1)^{2n}$$

$$= 1$$

$$\left[\frac{1}{1-x} = 1 + x + x^2 + \cdots \right.$$

$$(1 \underbrace{- 4x}_{})^{-1/2}$$

$$\alpha = -\frac{1}{2}$$

$$\boxed{y = -4x}$$

$$(1-y)^{-1/2} = \binom{-\frac{1}{2}}{0} + \binom{-1/2}{1} y + \cdots + \binom{-1/2}{n} y^n + \cdots$$

$$(-1)^{2n} = 1 \qquad \binom{-\frac{1}{2}}{n} y^n = \binom{-1/2}{n} (-4x)^n$$

$$\boxed{(-1)^n \cancel{(-1)^n}} \quad \frac{\frac{1}{\textcircled{2}}\left(\cancel{\frac{3}{\textcircled{2}}}\right)\left(\cancel{\frac{5}{\textcircled{2}}}\right) \cdots \left(\frac{2n-1}{\textcircled{2}}\right)}{n!} (2\times2)^n$$

$$\frac{1 \overset{2}{\cdot} 3 \overset{4}{\cdot} 5 \overset{6}{\cdot} \cdots \overset{2n-2}{} (2n-1) \overset{2n}{}}{\cancel{2^n}\; n!\; n!} \cancel{2^n}\; 2^n\, \cancel{2\; n!}$$

$$1\ 2\ 3\ 4 \ldots n$$

$$\frac{2n!}{n!\;n!} = \binom{2n}{n}$$

$$\boxed{\begin{aligned}(1-4x)^{-1/2} &= 1+\binom{2}{1}x^1+\binom{4}{2}x^2+\cdots\\ &\quad+\binom{2n}{n}x^n+\cdots\end{aligned}}$$

$$h_1 = \binom{2n}{n}$$

$$1,\ 2,\ \binom{4}{2},\ \binom{6}{3},\ \cdots$$

$$(1+x)^{\alpha} = \binom{\alpha}{0}+\binom{\alpha}{1}x+\cdots+\binom{\alpha}{n}x^n+\cdots$$

$$(1+\underbrace{\gamma x}_{y})^{\overset{\alpha}{\overset{\|}{-k}}} = \binom{-k}{0}+\binom{-k}{1}\gamma x+\binom{-k}{2}\gamma^2x^2+\cdots+\binom{-k}{n}\gamma^n x^n+\cdots$$

$$(1 \underbrace{- rx}_{y})^{-k} = \binom{-k}{0} + \binom{-k}{1}(-r)x + \binom{-k}{2}(-r)^2 x^2 + \cdots + \binom{-k}{n}(-r)^n x^n + \cdots\cdots$$

$$\binom{-k}{n}(-r)^n \;\downarrow\; \binom{n+k-1}{n} r^n$$

$$\boxed{\begin{aligned} \alpha &= -k \\ y &= -rx \end{aligned}}$$

$$\binom{-k}{n}(-r)^n \longrightarrow (-1)^n \binom{n+k-1}{n} (-1)^n r^n$$

$$\cancel{(-1)^{2n}}\binom{n+k-1}{n}\gamma^{n}$$

$$\binom{n+k-1}{n}\gamma^{n}$$

$$(1-\gamma x)^{-k} = \cdots + \underline{\binom{n+k-1}{n}\underset{\overset{\uparrow}{1}}{\gamma^{n}}\,x^{n}}$$

$$+ \cdots$$

$$\Big\downarrow$$

$$\underline{= 1 + k\gamma x + \binom{k+1}{2}\gamma^{2}x^{2} + \cdots}$$

$h_0, h_1, h_2, \ldots, h_n, \ldots$

$$\boxed{e_1 + e_2 + \cdots + e_k = n}$$

$e_i \geq 0$

$$\boxed{h_n = \binom{n+k-1}{n}}$$

$$\boxed{h_0 = 1}$$

$$g(x) = 1 + h_1 x + h_2 x^2 + \cdots + \underline{h_n} x^n$$

$$\binom{n+k-1}{n} x^n$$

$$(1-x)^{-k} = \cdots + \underbrace{\binom{n+k-1}{n} x^n}_{} + \cdots$$

$$(1-x)^{-k} = \frac{1}{(1-x)^k}$$

$$\boxed{\frac{1}{(1-x)^k}} = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x}\right)\cdots\left(\frac{1}{1-x}\right)$$

$$g(x) = \left[1+x+x^2+\ \cdots\ \right]\left[1+x+x^2+\cdots\right]\cdots\left[1+x+x^2+x^3+\cdots\right]$$

$$\boxed{x^{e_1^{\checkmark}}\, x^{e_2^{\checkmark}} \cdots x^{e_k^{\checkmark}} = x^n} \Leftrightarrow \boxed{e_1+e_2+\cdots+e_k=n}$$

$$\frac{1}{1-x}\,\frac{1}{1-x}\cdots\frac{1}{1-x} = \left(\frac{1}{1-x}\right)^{k}$$

$$\frac{1}{1-x} = \left(1 + x + x^2 + - - - - - -\right)$$

$$\boxed{\left(1+x+x^2+\cdots+x^5\right)}\,\boxed{\left(1+x+x^2\right)}\left(1+x+x^2+\cdots+x^4\right)$$

$$= \left(\frac{1-x^6}{1-x}\right)\left(\frac{1-x^3}{1-x}\right)\left(\frac{1-x^5}{1-x}\right)$$

$$\downarrow$$

$$e_1 + e_2 + e_3 + e_4 = n$$

#apples #bananas #oranges pears

e_1 → "even"

e_2 → "odd"

e_3 $\quad 0 \le e_3 \le 4$

$e_4 \ge 1$ at least

apples $\quad 1 + x^2 + x^4 + x^6 + \cdots$

bananas $\quad x(1 + x^2 + x^4 + \cdots)$

$x + x^3 + x^5 + \cdots$

$1 + x + x^2 + x^3 + x^4$

$x(1 + x + x^2 + \cdots)$

$x + x^2 + x^3 + \cdots$

$$\frac{1}{(1-x^2)} \times \frac{x}{(1-x^2)} \times \frac{1-x^5}{1-x} \times \frac{x}{1-x}$$

$$x^{e_1 + e_2 + e_3 + e_4 = n}$$

$$\underline{1 + x^2 + x^4 + x^6 + \cdots}$$

$y = x^2$ $\quad 1 + y + y^2 + y^3 + \cdots$

$$= \frac{1}{1-y} = \underline{\underline{\frac{1}{1-x^2}}}$$

$$\boxed{\frac{x^2(1-x^5)}{(1-x^2)^2(1-x)^2}}$$

x^n

$$= \frac{x^2 - x^7}{(1-x^2)^2(1-x)^2}$$

$$\boxed{\frac{x^2}{(1-x^2)^2(1-x)^2}} - \frac{x^1}{(1-x^2)^2(1-x)^2}$$

$$\cancel{x^2}\left[\frac{1}{(1-x^2)^2(1-x)^2}\right] \leftarrow x^{n-2}$$

$$\frac{1}{(1-x^2)^2(1-x)^2} \qquad (1+x)^\alpha$$

$$f(x) = (1-x^2)^2$$

$$g(x) = (1-x)^2$$

$$h(x) = f(x).g(x)$$

$$h(x) = c_0 + c_1 x + \cdots + c_n x^n \cdots$$

$$c_n = a_0 b_n + \cdots +$$

$$\underbrace{\boxed{e_1} + \boxed{e_2} + \cdots + e_k} = n$$

$$\left(x + x^3 + x^5 + \cdots \cdots\right)\left(\underbrace{+ - + - + + -}\right)$$

$$\frac{x^7}{(1-x^2)^2(1-x)^2}$$

$$\begin{tikzcd} & \underbrace{x^n}_{} \arrow[d] \arrow[d, bend left] \\ \left[\frac{1}{(1-x^2)^2(1-x)^2}\right] & \underbrace{x^{n-7}}_{} \arrow[l] \end{tikzcd}$$

$$h(x) = f(x) \cdot g(x)$$

$$\boxed{(1+y)^\alpha} \longleftarrow \boxed{f(x) = \frac{1}{(1-x^2)^2}}$$

when $\alpha = -2$

$y = (-x^2)$

$$g(x) = \frac{1}{(1-x)^2}$$

←

$$f(x) = a_0, a_1, a_2, ----$$

$$f(x) = \boxed{a_0} + \underbrace{a_1 x} + a_2 x^2 + - - - -$$

$$g(x) = b_0 + b_1 x + b_2 x^2 + - + b_{n-1} x^{n-1} + b_n x^n$$

$$h(x) = f(x) \cdot g(x) = c_0 + c_1 x + c_2 x^2 + ---$$

$$c_n = \Big[\underline{\underbrace{a_0 b_n} + a_1 b_{n-1} + a_2 b_{n-2}}$$

$$\underline{+ \cdots + a_n b_0}\Big]$$

$$e_1 + e_2 + e_3 + e_4 = n$$

apples

bananas

e_1 ✓ $\to$ even $\to (1 + x^2 + x^4 + \cdots)$

$\times$

e_2 ✓ $\to$ multiple of 5 $\to (1 + x^5 + x^{10} + \cdots)$

$\times$

e_3 ✓ $\to 0 \le e_3 \le 4 \to (1 + x + \cdots + x^4)$

$\times$

e_4 ✓ $\to 0 \le e_4 \le 1 \to (1 + x)$

$$\frac{1}{(1-x^2)} \qquad \frac{1}{(1-x^5)} \qquad \frac{\cancel{1-x^5}}{1-x} \qquad (1+x)$$

$$\to \frac{1}{\cancel{(1+x)}(1-x)} \frac{\cancel{(1+x)}}{(1-x)}$$

$y = x^5$

$$\frac{1}{1-y} = 1 + y + y^2 + \cdots = 1 + x^5 + x^{10} + x^{15} + \cdots$$

$$\frac{1}{(1-x)^2}$$

coefficient of x^n

$$(1+y)^{\alpha} \quad \text{where} \quad \alpha = -2$$

$\uparrow$ $y = -x$

$$\binom{\alpha}{n} y^n$$

$$\frac{(-2)^{\underline{n}}}{n!} \qquad \boxed{\binom{-2}{n}}(-x)^n$$

$$\frac{1(-2)(-3)(-4)\cdots\cdots(-(n+1))}{n!}(-1)^n x^n$$

$\cancel{(-1)^{2n}}$

$$\cancel{(-1)^n}\cdot\frac{\cancel{2\cdot3\cdots n}(n+1)\cancel{(-1)^n}\ x^n}{\cancel{n!}}$$

$$\downarrow$$

$$\boxed{n+1} \longleftarrow \underline{(n+1)\,x^n}$$

$$\boxed{e_1} + \cdots + e_k = n$$

e_i is non-negative odd integer

$\hookrightarrow$ 1, 3, 5, 7, ---

$$(x+x^3+x^5+\cdots)(x+x^3+x^5+\cdots)\cdots(x+x^3+\cdots)$$

$$(x + x^3 + \cdots)^k$$

$$\left[x(1 + x^2 + x^4 + \cdots)\right]^k$$

$$= \boxed{x^k}\left(\frac{1}{1 - x^2}\right)^k$$

$$\boxed{x^{\underline{n-k}}}$$

$$\frac{1}{(1 - x^2)^k}$$

$$(1+y)^\alpha = - - - - \binom{\alpha}{n} y^n$$

where $\alpha = -k$, $y = -x^2$

$$\binom{-k}{n'}\left(-x^{2}\right)^{n'}$$

$$\binom{-k}{n'}(-1)^{n'}\underbrace{x^{\boxed{2n'}}}$$

$$\boxed{e_1+e_2+\cdots+e_{\underline{k}}=\boxed{n}}$$

$$2n' = \boxed{n-k}$$

$$\boxed{n' = \frac{n-k}{2}}$$

$$1 \quad = 1$$

$$\left.\begin{matrix}2\\1+1\end{matrix}\right\} \longrightarrow 2$$

$$\left.\begin{matrix}3\\1+2\\2+1\\1+1+1\end{matrix}\right\} 4=2^{2} \qquad 3$$

4

1 + 3

2 + 2

3 + 1

1 + 1 + 2

1 + 2 + 1

2 + 1 + 1

|

4

1 + 1 + 1 + 1

$8 = 2^3 = 4$

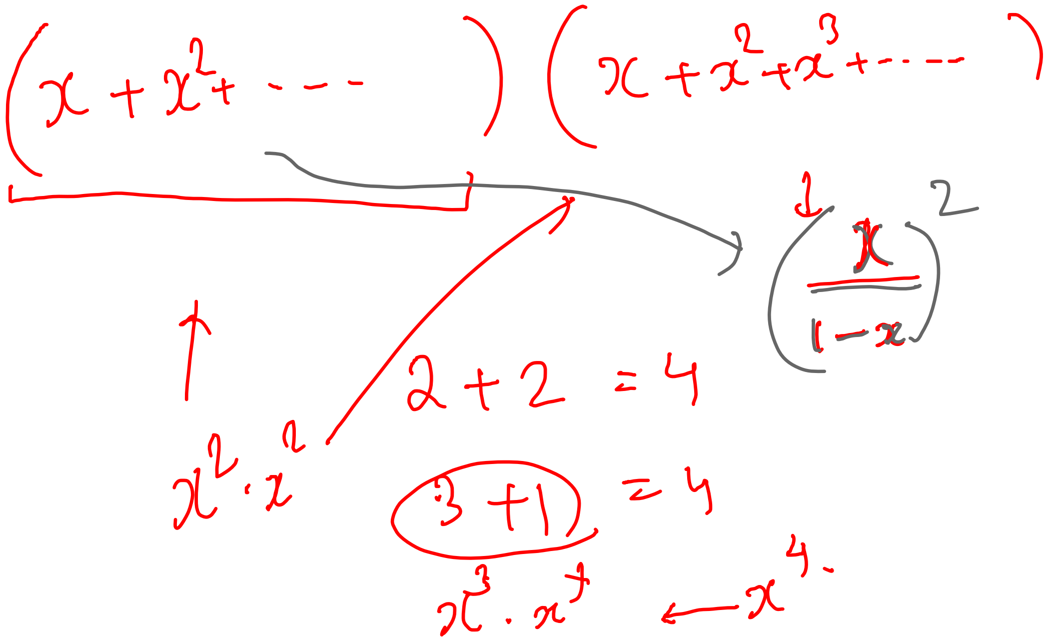
$\boxed{2^{n-1}}$ ↙

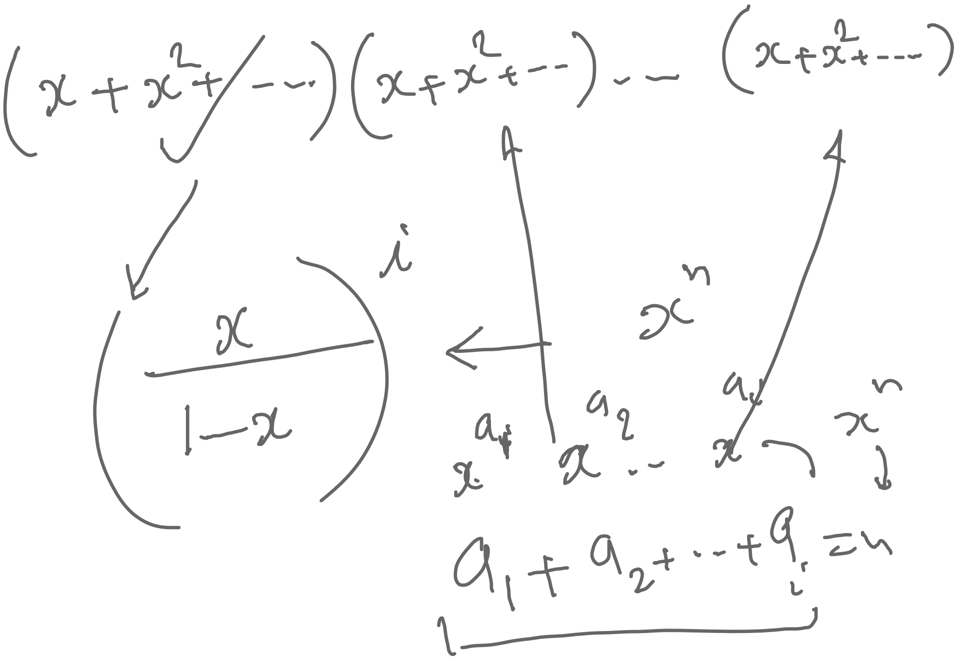
i summands

$i = 1$

x^n

$(x + x^2 + x^3 + - - - - x^n -)$


$(x + x^2 + \cdots)(x + x^2 + x^3 + \cdots)$
$\left(\frac{x}{1-x}\right)^2$
$2 + 2 = 4$
$3 + 1 = 4$
$x^2 \cdot x^2$
$x^3 \cdot x^1$
x^4


$(x + x^2 + \cdots)(x + x^2 + \cdots)\cdots(x + x^2 + \cdots)$
$\left(\frac{x}{1-x}\right)^i$
x^n
$x^{a_1} x^{a_2} \cdots x^{a_i}$
x^n
$a_1 + a_2 + \cdots + a_i = n$
1

$$\sum_{i=1}^{\infty}\left(\frac{x}{1-x}\right)^{i}$$

x^n

$$\left(x+x^2+x^3+\cdots\right)\left(x^{'}\right.$$

$$\left(\frac{x}{1-x}\right)^{\underline{n+1}} =$$

x^n

$$\frac{x}{1-x}=\left(x+x^2+\cdots\right)$$

$$\sum_{i=1}^{\infty}\left(\frac{x}{1-x}\right)^{i} \qquad \boxed{y=\frac{x}{1-x}}$$

$$\sum_{i=1}^{\infty} y^{i} = y+y^{2}+y^{3}+\cdots$$

$$= y\underbrace{\left(1+y+y^{2}+\cdots\right)}$$

$$= y\left(\frac{1}{1-y}\right)$$

$$= \frac{x}{\cancel{1-x}}\left(\frac{1}{(1-x)-\frac{x}{1-x}}\right)$$

$$\boxed{x^{n-1}}$$

$$= \boxed{\frac{x}{1-2x}} \rightarrow \boxed{\frac{1}{1-2x}}$$

$$\frac{1}{(1-2x)} \checkmark$$

$$1, 2, 2^2, 2^3, ---$$

$(n-1)$ th term $\rightarrow \boxed{2^{n-1}}$

$$\rightarrow 2\overset{\checkmark}{e}_1 + 3\overset{\checkmark}{e}_2 + \underbrace{4}\overset{\checkmark}{e}_{\underbrace{3}} + \underbrace{5}\overset{\checkmark}{e}_4 = n \}$$

$$\boxed{f_1 + f_2 + \underbrace{f_3} + f_4 = n}$$

$f_1 = 2e_1$

$f_2 = \underbrace{3e_2}$

$f_3 = \underbrace{4e_3}$

$f_4 = \underbrace{5e_4}$

$$(1+x^2+x^4+x^6+\cdots)(1+x^3+x^6+x^9+\cdots)$$

$$(1+x^4+x^8+x^{12}+\cdots)(1+x^5+x^{10}+x^{15}+\cdots)$$

$$\left[\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^4}\right)\left(\frac{1}{1-x^5}\right)\right]$$

$$h_n = 5h_{n-1} - 6h_{n-2} \qquad \left(\begin{array}{l} n \geq 2 \\ h_0 = 1 \\ h_2 = -2 \end{array}\right.$$

Step 1

$$\boxed{\underset{\uparrow\,2}{h_n} - \underbrace{5}\,\underbrace{h_{n-1}}_{1} + \underbrace{6}\,\underbrace{h_{n-2}}_{0} = 0}$$

① ← $h(x) = h_0 + h_1 x + h_2 x^2 + \cdots h_3 x^3$

+

② ← $-5x\,h(x) = \cdots + -5h_0 x + -5h_1 x^2 + \cdots -5h_2 x^3$

+

③ ← $6x^2\,h(x) = \cdots \cdots 6h_0 x^2 + 6h_1 x^3 + \cdots$

—

$$\left[1-5x+6x^2\right] h(x) = \overset{1}{\cancel{h_0}} + \left[\overset{-2}{\cancel{h_1}} - 5\overset{1}{\cancel{h_0}}\right]x + \overset{0}{\cancel{\left[h_2 - 5h_1 + 6h_0\right]}}x^2 + \cancel{\left(h_3 - 5h_2 + 6h_1\right)}x^3 + \cancel{\left(\cdots\right)}$$

$$\left(1-5x+6x^2\right)h(x) = 1 - 7x$$

$$h(x) = \frac{1-7x}{1-5x+6x^2}$$

$1 \quad -2$

$h_0, h_1, h_2, \cdots$

$h_n = 5h_{n-1} - 6h_{n-2}$

$n \geq 3$

$$h(x) = \boxed{\frac{1-7x}{1-5x+6x^2}} \longrightarrow$$

$$h_0 + h_1 x + h_2 x^2 + \cdots + \boxed{h_n} x^n + \cdots$$

$$h_n =$$

$$\frac{1-7x}{(1-2x)(1-3x)}$$

$$=$$

$$\boxed{\begin{matrix} h_0 = 1 \\ h_1 = -2 \end{matrix}} \qquad h_n = 5h_{n-1} - 6h_{n-2}$$

$-10-6$

$h_0, h_1, h_2, h_3, \ldots\ldots$

$1, -2, -16, \ldots$

$$\boxed{g(x) = 1 - 2x - 16x^2 + \cdots}$$

$$\underbrace{g(x)} = h_0 + h_1 x + \cdots + h_n x^n + \cdots$$

x^n

$$\boxed{\overset{3}{h_n} - 5\overset{2}{h_{n-1}} + 6\overset{1}{h_{n-2}} = 0} \quad n \geq 2$$

$$g(x) = h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \cdots$$

$$+$$

$$(-5x)\, g(x) = -5h_0 x - 5h_1 x^2 - 5h_2 x^3 + \cdots$$

$$+$$

$$(6x^2)\, g(x) = 6h_0 x^2 + 6h_1 x^3 + \cdots$$

$$g(x)[1 - 5x + 6x^2] = \overset{1}{h_0} + [\overset{2}{h_1} - 5\overset{1}{h_0}]x + 0x^2 + 0x^3 + \cdots$$

$$g(x)[1 - 5x + 6x^2] = 1 - 7x$$

$$\boxed{g(x) = \frac{1 - 7x}{1 - 5x + 6x^2}}$$

$$\overset{\overset{1}{\downarrow}}{h_0}, \overset{\overset{-2}{\downarrow}}{h_1}, h_2, \ldots$$

Satisfying the equation $h_n - 5h_{n-1} + 6h_{n-2} = 0$

$$\boxed{\frac{1-7x}{1-5x+6x^2} = \frac{5}{1-2x} - \frac{4}{1-3x}}$$

$$1-7x = c_1(1-3x) + c_2(1-2x)$$

$$= \underbrace{(c_1+c_2)} + \underbrace{(-3c_1-2c_2)}x$$

$$c_1 + c_2 = 1$$

$$-3c_1 - 2c_2 = -7$$

$c_1 = 5 \qquad c_2 = -4$ ✓

$$\boxed{\frac{5}{1-2x}} - \frac{4}{1-3x}$$

$$5(1-2x)^{-1} = 5\sum_{k=0}^{\infty}\binom{-1}{k}(-2x)^{k}$$

$$= 5\sum_{k=0}^{\infty} 2^{k} x^{k}$$

$$\boxed{5\cdot 2^{n}} \leftarrow x^{n}$$

$$\frac{-4}{1-3x} = (-4)\sum_{k=0}^{\infty} 3^{k} x^{k}$$

$$x^{n} \rightarrow \boxed{-4\cdot 3^{n}}$$

$$\boxed{5\cdot 2^n - 4\cdot 3^n} \leftarrow x^n$$

$$\underbrace{\overset{k}{h_n} + a_1 \overset{k-1}{h_{n-1}} + a_2 \overset{k-2}{h_{n-2}} + \cdots + \boxed{a_k} \overset{0}{h_{n-k}} = 0} \quad \boxed{n \geq k}$$

$$\begin{aligned}
1\cdot g(x) &= h_0 + h_1 x + h_2 x^2 + \cdots + h_k x^k + \cdots \\
(a_1 x)\, g(x) &= \cdots + (a_1 h_0) x + a_1 h_1 x^2 + \cdots + a_1 h_{k-1} x^k + \cdots \\
(a_2 x^2)\, g(x) &= \cdots\cdots\cdots\ a_2 h_0 x^2 + \cdots + a_2 h_{k-2} x^k + \cdots \\
&\ \vdots \\
(a_k x^k)\, g(x) &= \cdots\cdots\cdots + a_k h_0 x^k + \cdots
\end{aligned}$$

$$\rightarrow g(x)\left[1 + a_1 x + a_2 x^2 + \cdots + a_k x^k\right] = \boxed{h_0 + (h_1 + a_1 h_0)x + \cdots}$$

$$g(x) \cdot \boxed{q(x)} = p(x)$$

$$g(x) = \frac{p(x)}{q(x)}$$

$$h_n \overset{a_1=1}{+} h_{n-1} \underbrace{\overset{a_2}{-16}} h_{n-2} + \underbrace{\overset{a_3}{20}} h_{n-3} = 0$$

$$\underbrace{n \geq 3}$$

$$\boxed{h_0 = 0, \quad h_1 = 1, \quad h_2 = -1} \leftarrow \qquad \boxed{x} = \underline{p(x)}$$

$$g(x) = \frac{p(x)}{q(x)} \quad \boxed{q(x) = 1 + x - 16x^2 + 20x^3} \checkmark$$

$$g(x) = \frac{P(x)}{q(x)} = \left[\frac{x}{1 + x - 16x^2 + 20x^3} \right]$$

$$1 + x - 16x^2 + 20x^3 = (1-2x)^2 (1+5x)$$

$$= \frac{c_1}{(1-2x)} \checkmark + \frac{c_2}{(1-2x)^2} + \frac{c_3}{(1+5x)}$$

$$\begin{aligned} x = {} & c_1 (1-2x)(1+5x) \\ & + c_2 (1+5x) \\ & + c_3 (1-2x)^2 \end{aligned}$$

$$C_2(1-2x)^{-2} = \frac{C_2}{(1-\underbrace{2x})^2} = \sum_{k=0}^{\infty}\binom{-2}{k}(-2x)^k$$

$$= \sum_{k=0}^{\infty}\binom{k+1}{k}(-1)^k(-2)^k x^k$$

$$= C_2\sum_{k=0}^{\infty}(k+1)\,2^k\,x^k$$

x^n $\quad \underbrace{C_2}(n+1)\,2^n$ ✓

$$h_n = \underbrace{\overset{c_1}{\left(\frac{-2}{49}\right)}}\,\underset{\smile}{2}^{n} + \overset{c_2}{\left(\frac{1}{49}\right)}(n+1)\,\underset{c_3}{2}^{n} \; \underbrace{-\left(\frac{5}{49}\right)}\,\underbrace{(-5)}^{n}$$

$$g(x) = \frac{p(x)}{q(x)} \rightarrow$$

$$\frac{c_1}{(1-2x)} + \frac{c_2}{(1-2x)^2} + \frac{c_3}{(1-5x)}$$

$$\boxed{q(x)} = (1-\underbrace{2}x)^2(1+\underbrace{5}x)$$

$$h_n + h_{n-1} - 16h_{n-2} + 20h_{n-3} = 0$$

$$\boxed{h_n = x^n}$$

$$\overset{x^3}{\cancel{x^n}} + \overset{x^2}{\cancel{x^{n-1}}} - 16\overset{x}{\cancel{x^{n-2}}} + 20\cancel{x^{n-3}} = 0$$

$$\boxed{x^3 + x^2 - 16x + 20 = 0}$$ ✓

$$q_1 \quad q_2 \quad \cdots \quad q_k$$

$$r(x) = (x-q_1)(x-q_2)\cdots(x-q_k)$$

$$r\left(\frac{1}{x}\right) = \left(\frac{1}{x}-q_1\right)\left(\frac{1}{x}-q_2\right)\cdots\left(\frac{1}{x}-q_k\right)$$

$$= \left[\frac{(1-q_1x)(1-q_2x)\cdots(1-q_kx)}{x^k}\right]$$

$$\boxed{r(x) = x^{k} + a_1 x^{k-1} + a_2 x^{k-2} + \cdots + a_k}$$

$$\boxed{q(x) = 1 + a_1 x + a_2 x^{2} + \cdots + a_k x^{k}}$$

$$\boxed{q(x) = x^{k}\left(r\left(\frac{1}{x}\right)\right)} \quad \checkmark$$

$$\boxed{h_n + a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k} = 0}$$

$$r(x) = 0$$

$$q_1, \ldots q_k$$

$$r(x) = (x - q_1)(x - q_2) \cdots (x - q_k)$$

$$r\left(\frac{1}{x}\right) = \frac{1}{x^{k}}\left[1 - q_1 x\right]\left[1 - q_2 x\right] \cdots \left[1 - q_k x\right]$$

$$q(x) = (1 - \underbrace{q_1} x)(1 - \underbrace{q_2} x) \cdots (1 - \underbrace{q_k} x)$$

$$\boxed{g(x) = \frac{p(x)}{q(x)} = \frac{c_1}{(1 - \underbrace{q_1} x)} + \frac{c_1}{(1 - q_2 x)} + \cdots + \frac{c_k}{(1 - q_k x}}$$

$$g(x) = \frac{p(x)}{q(x)}$$

p(x) is of degree $< k$

q(x) is of degree $= k$, and its constant term $\neq 0$

$$\underbrace{h_0, h_1, h_2, h_3,} \; - - - -$$

$$\left[h_n = \underbrace{a_1} h_{n-1} + \underbrace{a_2} h_{n-2} + \cdots + \underbrace{a_k} h_{n-k} \right]$$

$$\boxed{n \geq k}$$

$$\underbrace{g(x)} = \underbrace{h_0} + \underbrace{h_1} x + \underbrace{h_2} x^2 + \underbrace{h_3} x^3 + \cdots$$

$$g(x) = \frac{p(x)}{q(x)} \; ;$$

$$p(x) = q(x)\left[h_0 + h_1 x + h_2 x^2 + \cdots\right]$$

$$\left[d_0 + d_1 x + d_2 x^2 + \cdots + d_{k-1} x^{k-1}\right] = \left[b_0 + b_1 x + b_2 x^2 + \cdots + b_k x^k\right] \times \left[h_0 + h_1 x + h_2 x^2 + h_3 x^3 + \cdots\right]$$

h_0
h_1
h_2
$\vdots$
h_{k-1}

✓

$$b h_0 = d_0 \quad ①$$

$$b_1 h_0 + b_0 h_1 = d_1 \quad ②$$

$$b_0 h_2 + b_1 h_1 + b_2 h_0 = d_2 \quad ③$$

$$\vdots$$

$$b_0 h_{k-1} + b_1 h_{k-2} + \cdots + b_{k-1} h_0 = d_{k-1} \quad (k-1)$$

$$b_0 h_k + b_1 h_{k-1} + \cdots\cdots + b_k h_0 = 0$$

$$b_0 h_{k+1} + b_1' h_k + \cdots\cdots + b_k h_1 = 0$$

$n \geq k$

$$\boxed{h_n + \frac{b_1}{b_0} h_{n-1} + \frac{b_2}{b_0} h_{n-2} \cdots + b_k/b_0\, h_{n-k} = 0}$$

$$\boxed{h_n + \boxed{\frac{b_1}{b_0}} h_{n-1} + \boxed{\frac{b_2}{b_2}} h_{n-2} + \cdots + \boxed{\frac{b_k}{b_0}} h_{n-k} = 0}$$

$\boxed{n \geq k}$

$h_0, h_1, h_2, \dots, h_{k-1}, h_k, \dots$ ✓

$$\boxed{g(x) = \frac{p(x)}{q(x)}}$$

$\underline{a(n,r)} \leftarrow \quad \underbrace{1, 2, \overset{r}{3}, \dots, n-1}\ \cancel{n}$

$$a(n,r) = a(n,r-1) + a(n-1,r)$$

$\boxed{n \geq 0}$ $\quad \boxed{r \geq 1}$

$$a(n,1) = \overset{1}{\boxed{a(n,0)}} + \underline{a(n-1,1)}$$

$$\boxed{a(n,1) = n}$$

$$\boxed{a(n,1) = \boxed{a(n,0)} + \underbrace{a(n-1,1)}}$$

$$\boxed{r \geq 1}$$

$$a(n,0) = 1$$

$$\boxed{\begin{array}{r} a(n,r) = a(n,r-1) + a(n-1,r) \\ r \geq 1 \end{array}}$$

$$\underset{\textcircled{1}}{a(\underline{n},0)},\ \underset{\substack{\downarrow \\ n}}{a(\underline{n},1)},\ a(\underline{n},2),\ a(\underline{n},3),\ \ldots$$

$$f_n(x) = \sum_{r=0}^{\infty} a(n,r)\,x^r$$

$$f_0(x) = \sum_{r=1}^{\infty} a(0,r)\,x^r$$

$$= \underline{\overset{\overset{1}{\downarrow}}{a(0,0)}} + \cancel{a(0,1)}\,x + \cancel{a(0,2)}\,x^2 \cancel{+\cdots}$$

$$\boxed{0 = a(0,r),}\quad r>0$$

$$f_0(x) = 1$$

$$a(n,r) = a(n-1,r) + a(n,r-1)$$

$$r \geqslant 1$$

$$n \geqslant 1$$

$$\boxed{\sum_{r=1}^{\infty} a(n,r)\, x^{r}} = x \sum_{r'=0}^{\infty} a\Big(n, \overset{\downarrow r'}{\boxed{r-1}}\Big)\, x^{\boxed{r-1} \leftarrow r'} + \sum_{r=1}^{\infty} a(n-1,r)\, x^{r}$$

$$f_n(x) - \cancel{a(n,0)} = x\, f_n(x) + f_{n-1}(x) - \cancel{a(n-1,0)}$$

$$(1-x)\,f_n(x) = f_{n-1}(x)$$

$$f_n(x) = (1-x)^{-1}\,f_{n-1}(x)$$

$$= \frac{f_{n-1}(x)}{(1-x)} = \frac{f_{n-2}(x)}{(1-x)^2}$$

$$= \frac{f_0(x)}{(1-x)^n} = \frac{1}{(1-x)^n}$$

$$f_n(x) = \frac{1}{(1-x)^n}$$

$$\boxed{a(n,r)}$$

we look for the coefficient of x^n in $f_n(x)$

$$f_n(x) = \frac{1}{(1-x)^n} = (1-x)^{-n}$$

$$\binom{-n}{r} = \underline{\underline{\binom{r+n-1}{r}}}$$

$f_n(x)$

$$\left[\binom{n}{r} = \binom{n-1}{\underbrace{r-1}} + \binom{n-1}{r}\right] \longleftarrow \boxed{\frac{r \geq 1}{n \geq 1}}$$

$$\binom{n}{0} = 1 \qquad \boxed{\binom{n}{r} = 0 \;, \; r > n}$$

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \binom{n}{3}, \; - - - -$$

$$\boxed{f_n(x) = \sum_{r=0}^{\infty} \binom{n}{r} x^r}$$

$$f_0(x) = \underline{\binom{0}{0}}^{=1} + \cancel{\binom{\cancel{0}}{1} x} + \cancel{\binom{0}{2} x^2}$$

$$\boxed{\sum_{r=1}^{\infty}\binom{n}{r}x^{r}} = x\sum_{r'=0}^{\infty}\binom{n-1}{\boxed{r-1}\,r'}x^{\overset{r'}{\boxed{r-1}}} + \sum_{r=1}^{\infty}\binom{n-1}{r}x^{r}$$

$$f_n(x) - \cancel{\binom{n}{0}} = x\, f_{n-1}(x) + f_{n-1}(x) - \cancel{\binom{n-1}{0}}$$

$$f_n(x) = (1+x)\, f_{n-1}(x)$$

$$= (1+x)^2 f_{n-2}(x)$$

$$\vdots$$

$$= (1+x)^n f_0(x) = \underline{\underline{(1+x)^n}}$$

$$\left.\begin{aligned} a_{n+1} &= 2a_n + b_n \\ b_{n+1} &= a_n + b_n \end{aligned}\right\} \quad \underline{\underline{n \geq 0}}$$

$$a(x) = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

$$b(x) = \sum_{r=0}^{\infty} b_r x^r$$

$$\sum_{n=0}^{\infty} a_{n+1} x^{n+1} = x \sum_{n=0}^{\infty} 2 a_n x^n + x \sum_{n=0}^{\infty} b_n x^n$$

①— $$\boxed{a(x) - 1 = 2x\, a(x) + x\, b(x)}$$

$$\sum_{n=0}^{\infty} b_{n+1} x^{n+1} = \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} b_n x^{n+1}$$

$$b(x) - 0 = \underbrace{x\,a(x)}_{} + x\,b(x)$$

$$x \quad a(x) + (x-1)\,b(x) = 0 \quad -①$$

$$(1-2x)\,a(x) - x \quad b(x) \underline{= 1 \quad -①}$$

$$\sum_{n=0}^{\infty} a_{n+1} x^{n+1} = x\sum_{n=0}^{\infty} 2 a_n x^n + x\sum_{n=0}^{\infty} b_n x^n$$

$$\sum_{n=0}^{\infty} b_{n+1} x^{n+1} = \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} b_n x^{n+1}$$

$$\boxed{a(x) - 1 = 2x\, a(x) + x\, b(x)}$$

$$\boxed{(1-2x)\,a(x) - x\, b(x) = 1} \quad \text{—①}$$

$$b(x) - 0 = x\, a(x) + x\, b(x)$$

$$\boxed{x\, a(x) + (x-1)\, b(x) = 0} \quad \text{—①}$$

✓

$$a_0, a_1, a_2, \cdots$$

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$\left[a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots \right]$$

$$\boxed{1, 1, 1, 1, 1, \cdots}$$

$$1 + x + x^2 + \cdots = \frac{1}{1-x}$$

$$\boxed{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \underline{\underline{e^x}}}$$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \cdots$$

$$\to \underline{\binom{n}{0}, \binom{n}{1}, \ldots\ldots, \binom{n}{i}, \ldots\ldots}$$

$$\boxed{(1+x)^n} = a_0 + a_1 x + \frac{{}^nP_2\, x^2}{2!} + \cdots$$

$$\binom{n}{r},\ x^r$$

$$\boxed{a_r = {}^nP_r}$$

$$\frac{n!}{(n-r)!},\ \frac{x^r}{r!}$$

$$(1+x)^n$$

$${}^nP_0,\ {}^nP_1,\ {}^nP_2, \ldots\ldots$$

ENGINE

E | N | G | I

$\overset{\text{NN}}{}$

$$\overset{\text{EE}}{} \quad f(x) = \left(1 + x + \frac{x^2}{2!}\right)\left(1 + x + \frac{x^2}{2!}\right)(1+x)(1+x)$$

$\underline{\text{EE}}\ \underline{\text{NN}}$

$$\boxed{\cancel{\frac{x^4}{2!\,2!}}}\ \xrightarrow{5}$$

$$\frac{x^4}{2!\,2!} = \frac{4!}{\underbrace{2!\,2!}}\,\boxed{\frac{x^4}{4!}}$$

$$\frac{x^4}{4!}$$

$$\boxed{\frac{4!}{\underbrace{2!}\,\underbrace{2!}}}$$

$\underline{\boxed{EE}\,\boxed{NN}}$

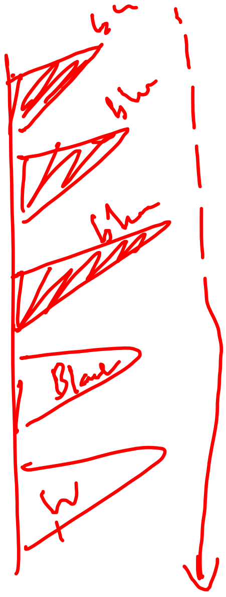
$$\frac{x^4}{4!}$$

$$\overset{E}{x}\quad \overset{NN}{\frac{x^2}{2!}}\quad \overset{I}{x}\ .\ 1$$

$\boxed{E\,\underbrace{NN}\,I}$

$$\frac{x^4}{2!} = \boxed{\frac{4!}{2!}}\left(\frac{x^4}{4!}\right)$$

$$\boxed{\frac{4!}{2!}}$$



✓ 48

12 W	12 black	12 blue	12 red

white ↓ red

$$\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^{12}}{12!}\right)\left(1+x+\cdots+\frac{x^{12}}{12!}\right)$$

blue black

$$\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^{12}}{12!}\right)\left(1+x+\cdots+\frac{x^{12}}{12}\right)$$

$$\frac{x^{k_1}}{k_1!}\cdot\frac{x^{k_2}}{k_2!}\cdot\frac{x^{k_3}}{k_3!}\cdot\frac{x^{k_4}}{k_4!}=\frac{x^{12}}{k!\cdot k_2!k_3!k_4!}\underbrace{k_1+k_2+\cdots k_3+k_4}_{=12}$$

$$\frac{12!}{k_1!\,k_2!\,\underbrace{k_3!}\,k_4!}\left(\frac{x^{12}}{12!}\right) \checkmark$$

blue | black

k_3 has to be even | k_4 has to be odd

W

$$\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^{12}}{12}+\cdots\right)\overset{\text{red}}{\left(1+x+\cdots+\frac{x^{12}}{12!}\right)}\cdot \text{black}$$

$$\longrightarrow \underbrace{\overset{\text{blue}}{\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots+\frac{x^{12}}{12!}\right)}}\underbrace{\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)}_{+\frac{x^{11}}{11!}}$$

$\boxed{x^{12}}$ ↙

$$\left[\left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^{12}}{12!}+\boxed{\frac{x^{13}}{13!}}^{\checkmark}+\cdots\cdots\right)\underbrace{\left(1+x+\frac{x^2}{2!}+\cdots\right)}\right.$$

$$\underbrace{\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)}\left(x+\frac{x^3}{3!}+\cdots\right)$$

$$=\left[e^x\cdot e^x\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^x-e^{-x}}{2}\right)\right]$$

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$$

$$e^{-x}=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\cdots$$

$$\boxed{\frac{e^x-e^{-x}}{2}}=\underbrace{\left[x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right]}$$

$$\frac{e^{2x}}{4}\left[e^{x}+e^{-x}\right]\left[e^{x}-e^{-x}\right]$$

$$=\frac{e^{2x}}{4}\left[e^{2x}-e^{-2x}\right]$$

$$=\frac{1}{4}\left[e^{4x}-1\right]$$

$$=\frac{1}{4}\left[\left(\cancel{1}+4x+\frac{(4x)^2}{2!}+\frac{(4x)^3}{3!}+\cdots\right)-\cancel{1}\right]$$

$$=\frac{1}{4}\left[4x+\frac{4^2x^2}{2!}+\frac{4^3x^3}{3!}+\cdots\right]$$

$$\frac{\boxed{x^{12}}}{12!} \qquad \boxed{\frac{1}{4}\cdot 4^{12}}=\boxed{4^{11}} \qquad \frac{\boxed{4^{12}}\,x^{12}}{12!}$$

$$\left(1 + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)\underbrace{\left(1 + x + \frac{x^2}{2!} + \cdots\right)^3}$$

$$\left[e^x - x - \frac{x^2}{2!}\right] e^{3x}$$

$$f(x) \longrightarrow \left[\underline{e^{4x}} - \underbrace{x e^{3x}} - \underbrace{\frac{x^2}{2} e^{3x}}\right]$$

$$e^{4x} \longrightarrow 1 + 4x + \frac{(4x)^2}{2!} + \cdots + 4^{12}\frac{x^{12}}{12!}$$

$$\boxed{4^{12}} \quad - x e^{3x} \qquad \boxed{x}\left(1 + 3x + \frac{(3x)^2}{2!} + \cdots\right.$$

$$= \left(x + 3x^2 + \frac{\underline{3^2} x^3}{2!} + \cdots\right.$$

$$- 12 \cdot 3^{11} \boxed{\frac{x^{12}}{12!}} \qquad = \frac{\cancel{(12)} \cdot 3^{11}}{\cancel{11!}} \boxed{\frac{x^{12}}{12!}}$$

$$-\frac{x^2}{\textcircled{2}}e^{3x} \rightarrow \frac{x^2}{2}\left[x^2 + 3x^3 + \frac{3^2x^4}{2!} + \cdots\right]$$

$$(11\cdot 12)\,\cancel{12!}\,3^{10}\,\frac{x^{12}}{\cancel{10!}\cdot 12!} \qquad \frac{x^{12}}{12}$$

$$-\frac{11\cdot\overset{6}{\cancel{12}}\cdot 3^{10}}{\cancel{2}}\,\boxed{\frac{x^{12}}{12!}}$$

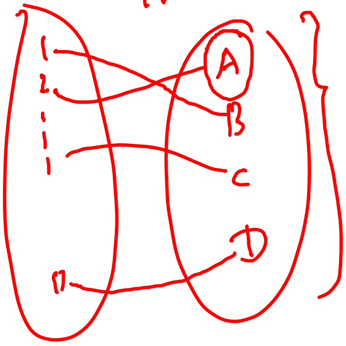
A B C D

11

[A B B . - - - .]

1 2 3 4 . . . 11

$[11] \rightarrow \{A, B, C, D\}$



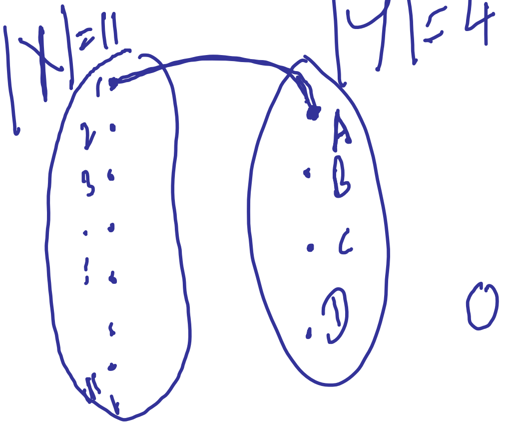
onto function

$$f(x) = \left[x + \frac{x^2}{2!} + \cdots + \frac{x^{11}}{11!} + \frac{x^{12}}{12!} \cdots\right]^4$$

A, B, C, D

A B B A A D

1 2 3 4 11



AADBB....

Onto function

A B C D

$$\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)\left(x+\frac{x^2}{2!}+\cdots\right)\left(x+\frac{x^2}{2!}+\cdots\right)\left(x+\frac{x^2}{2!}+\cdots\right)$$

$$\frac{x^{k_1}}{k_1!}\cdot\frac{x^{k_2}}{k_2!}\cdot\frac{x^{k_3}}{k_3!}\cdot\frac{x^{k_4}}{k_4!}\to\frac{x^{11}}{k_1!\cdot k_2!\,k_3!\,k_4!}$$

$$\boxed{k_1+k_2+k_3+k_4=11}$$

$$\to x^{11}\to\frac{x^{11}}{11!}$$

$$(11!)\frac{x^{11}}{k_1!\,k_2!\,k_3!\,k_4!\quad 11!}$$

$$\frac{A^{k_1}\;B^{k_2}\;C^{k_3}\;D^{k_4}}{11!}$$

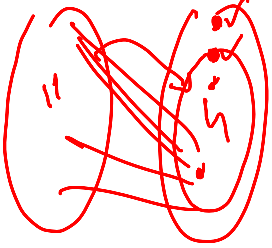
$$\frac{11!}{k_1!\,k_2!\,k_3!\,k_4!}$$

$$\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^4$$

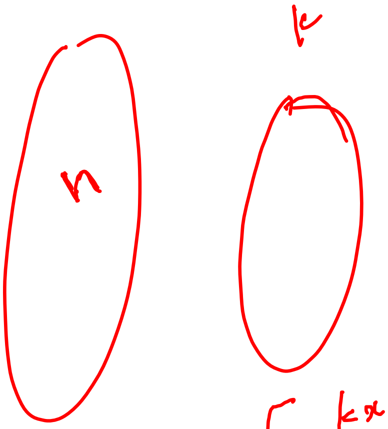
$$\left(e^x-1\right)^4=\underline{e^{4x}}-\binom{4}{1}e^{3x}+\boxed{\binom{4}{2}e^{2x}}-\binom{4}{3}e^{x}+\underline{1}$$

$$\frac{x^{11}}{11!}$$

$$e^{4x}=1+4x+\frac{(4x)^2}{2!}+\cdots+\frac{(4x)^{11}}{11!}+\cdots$$



$$\boxed{4^{11}-\binom{4}{1}3^{11}+\binom{4}{2}2^{11}-\binom{4}{3}1^{11}}$$



$$(e^x-1)^k \longrightarrow$$

$$\swarrow \qquad \boxed{x^n}$$

$$\left[\begin{array}{l} e^{kx} - \binom{k}{1} e^{(k-1)x} + \binom{k}{2} e^{(k-2)x} + \cdots \\ \downarrow \, k^n - \binom{k}{1}(k-1)^n + \binom{k}{2}(k-2)^n \cdots \end{array}\right.$$

$$\boxed{n = \underbrace{x_1\cdot 1} + \underbrace{x_2\cdot 2} + \underbrace{x_3\cdot 3} + \cdots + x_n\cdot n}$$

$$\boxed{p(n)}$$

$$\boxed{p(0) = 1}$$

$$p(0),\ p(1),\ p(2),\ p(3),\ \ldots,\ p(n),\ \ldots$$

$$\underline{n = x_1 \cdot \underbrace{1} + x_2 \cdot \underbrace{2} + \cdots + x_n \cdot \underbrace{n}}$$

$$6 = \overrightarrow{3+2+1}\ \checkmark$$

$p(n)$

$$\left(1 + x + x^2 + \cdots \cdots\right)\left(1 + x^2 + x^4 + x^6 + \cdots\right)$$

$$\left(1 + x^3 + x^6 + x^9 + \cdots\right)\left(1 + x^4 + x^8 + x^{12} + \cdots\right)$$

$$\left(1 + x^n + x^{2n} \cdots\right)\left(1 + \underbrace{x^{n+1}} + \cdots\right)$$

$$\left(1 + x + \cdots\right)\left(1 + x^2 + \cdots\cdots\right)\left(1 + x^3 + x^6 + \cdots\right)$$

$$\boxed{= \prod_{i=1}^{\infty} \frac{1}{(1-x^{i})}}$$

$$P(0),\ P(1),\ P(2),\ P(3),\ \text{- - - - -}\ \overset{\downarrow}{\boxed{P(n)}}\ \text{- -}$$

$$\boxed{P(n)}$$

$n=10$

$k=4$

$$5 \quad 2 \quad 2 \quad 1$$

$$\boxed{x_1 + x_2 + x_3 + x_4 = n}$$

$$\binom{n+k-1}{n}$$

$$\boxed{x_k \geq 0}$$

$$x_1 \geq x_2 \geq x_3 \geq x_4 \geq 1$$

$$x_1 + x_2 + x_3 + x_4 = n$$

$$\boxed{x_i \geq 1}$$

$$\boxed{y_1 + y_2 + y_3 + y_4 = n - 4}$$

k

$$x_1 + \cdots + x_k = n$$

$$\boxed{x_i \geq 1}$$

$$\boxed{x_1 \geq x_2 \geq \cdots \geq x_k}$$

$$\boxed{P_k(n)}$$

$$\underbrace{P_k(n)} = \sum_{s=1}^{k} P_s(n-k)$$

$$x_1 + x_2 + \cdots + x_k = n$$

$$x_1 \geq x_2 \geq \cdots \geq x_k \geq 1$$

$$\boxed{y_i = x_i - 1}$$

$$\underbrace{y_1 + y_2 + \cdots + \boxed{y_k}} = n-k$$

$$\boxed{y_1 \geq y_2 \geq \cdots \geq y_k \geq 0}$$

$$\boxed{P_k(n) = \sum_{s=1}^{k} P_s(\underbrace{n-k})} \checkmark$$

$$= P_k(n-k) + P_{k-1}(n-k) + P_{k-2}(n-k) + \vdots\ P_1(n-k = P_k(n)$$

$$\boxed{P_1(n) = 1}$$

$$\boxed{y_1 = n-k}$$

$$\boxed{P_1(n) = \sum_{s=1} P(n-1) = P_1(n-1) = P_1(n-2) \; = \boxed{P_1(1) = 1}}$$

$$\boxed{P_1(n) = 1}$$

$$\boxed{P_k(n) = \boxed{P_{k-1}(n-1)} + P_k(n-k)}$$ ✓

$$n = x_1 + x_2 + \cdots + x_k$$

$$x_1 \geq x_2 \geq \cdots \geq x_k \geq 2$$

$$y_i = x_i - 1$$

$$n-k = y_1 + y_2 + \cdots + y_k$$

$$\boxed{y_1 \geq y_2 \geq \cdots \geq y_k \geq 1}$$

$$P_2(n) = \boxed{P_1(n-1)} + P_2(n-2)$$

$$= 1 + P_2(n-2)$$

$$= \underbrace{1 + 1} + P_2(n-4)$$

$$\downarrow$$

$$\left\lfloor \frac{n}{2} \right\rfloor$$

$$n = x_1 + x_2$$

$\uparrow$ where $x_1 \geq x_2 \geq 1$

$$\left\lfloor \frac{n}{2} \right\rfloor \qquad \left\lfloor \frac{n}{2} \right\rfloor$$

$$(k!)\, P_k(n) \geq \boxed{\binom{n-1}{k-1}}$$ ✓

$(k!) \curvearrowright$ $\boxed{x_1 + x_2 + x_3 + \dots + x_k = n}$ ✓✓

$\left\{ \boxed{x_1 \geq x_2 \geq \dots \geq x_k \geq 1} \right.$ ✓✓

$\boxed{y_i \geq 1}$ $\left[\underline{y_1 + y_2 + \dots + y_1 = n} \right.$ ✓

$$y_1 + \dots + y_k = n$$

$$\boxed{y_i \geq 1}$$

$$\downarrow$$

$$\boxed{z_1 + \dots + z_k = n-k}$$

by defining

$z_i = y_i - 1 \curvearrowright z_i \geq 0$

$$\binom{n-\cancel{k}+k-1}{k-1}$$

$$= \binom{n-1}{k-1}$$

$$(k!)\, p_k(n) \le \binom{n+\frac{k(k-1)}{2}-1}{k-1}$$

$$x_1 + x_2 + \cdots + x_k = n.$$

$$x_i \geq 1$$

$$\boxed{y_1 + y_2 + \cdots + y_k = n + \frac{k(k-1)}{2}}$$

$$y_i = x_i + k - i$$

$$\left.\begin{aligned} y_1 &= x_1 + k - 1 \\ y_2 &= x_2 + k - 2 \\ &\vdots \\ y_k &= x_k + k - k \end{aligned}\right\}$$

$$\left.\begin{matrix} k-1 \\ k-2 \\ \vdots \\ + \\ 1 \end{matrix}\right\}$$

$$\boxed{\frac{k(k-1)}{2}}$$

$$y_1 + y_2 + \cdots + y_k = n + \frac{k(k-1)}{2}$$

$$\boxed{y_i \geq 1}$$

$$\boxed{k!\, P_k(n) \leq \binom{n + \frac{k(k-1)}{2} - 1}{k-1}}$$

$$\boxed{\begin{aligned} y_i &= x_i + k - i \\ y_j &= x_j + k - j \end{aligned}}$$

$\underline{i < j}$

$$x_i + k - i = x_j + k - j$$

$$\underline{x_i - x_j = i - j} \quad \propto$$

$$\underbrace{\binom{n-1}{k-1} \leq (k!)\, P_k(n)} \leq \binom{n + \frac{k(k-1)}{2} - 1}{k-1}$$

$$\frac{\underline{(n-1)(n-2)\cdots(n-1-(k-1)+1)}}{(k-1)!\quad k!} \sim n^{k-1} \leq P_k(n)$$

as $n \to \infty$

$\boxed{\text{fixed } k}$

$$\frac{n^{k-1}}{k!\,(n-k)!}$$

$$P_k(n) \leq \frac{1}{k!} \binom{n + \frac{k(k-1)}{2} - 1}{k-1}$$

$$\leq \frac{1}{k!} \frac{n^{k-1}}{(k-1)!}$$

$$\boxed{P_k(n) \sim \frac{n^{k-1}}{k!\,(k-1)!}}$$

$n \to \infty$

fixed k

$$\begin{aligned}10 &= 10\\ &= 9+1\\ &= 8+2\\ &= 7+3\\ &= 6+4\\ &= \cancel{\boxed{5+5}}\end{aligned}\qquad \begin{aligned}10 &= \boxed{8+\boxed{\cancel{1}+\cancel{1}}}\\ &= \underline{7+2+1}\end{aligned}$$

$$\underline{P_D(10)} \qquad P_D(n)$$

$$1 \quad P_D(n)$$

$$\|$$

$$P_D(0),\ P_D(1),\ P_D(2),\ \text{- - - -}$$

$$\sum_{i=0}^{\infty} P_D(i)\,x^i = \prod_{i=1}^{\infty}\left(1+x^i\right)$$

$$\underline{(1+x)}\,(1+x^2)(1+x^3)(1+x^4)$$

$P_0(n)$

$$10 = \cancel{10}$$

$$10 = 9 + 1$$

$$= \cancel{8} + \cancel{2}$$

$$= 7 + \boxed{2} + 1$$

$$= 7 + 1 + 1 + 1$$

$$= 5 + 5$$

$$10 = \underbrace{1 + 1 + 1 + \cdots + 1}_{10}$$ ✓

$P_0(10)$ $P_0(n)$

	$P_D(n)$	$P_0(n)$
n	-	-

$P_0(0) = 1$

→ $\underbrace{P_0(0),\ P_0(1),\ P_0(2),\ ----}$

→ $P_D(0),\ P_D(1),\ P_D(2),\ ---$

↪ $\boxed{P_D(n) = P_0(n)}$

$$(1+x+x^2+\cdots)(1+x^3+x^6+x^9+\cdots)$$

$$(1+x^5+x^{10}+x^{15}+\cdots)\cdots$$

$$P_0(x) = \left[\frac{1}{1-x}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^5}\cdots\cdots\right]$$

$$P_D(x) = (1+x)(1+x^2)(1+x^3)\cdots\cdots$$

$$1+x = \frac{1-x^2}{1-x}$$

$$1+x^2 = \frac{1-x^4}{1-x^2}$$

$$1+x^3 = \frac{1-x^6}{1-x^3}$$

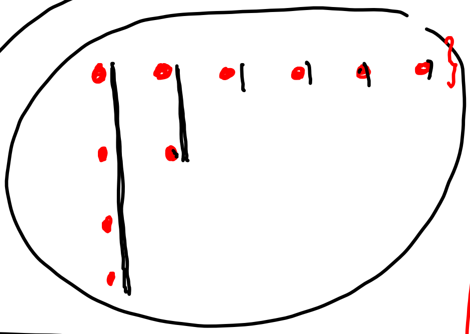
$$\frac{\cancel{(1-x^2)}}{(1-x)}\cdot\frac{\cancel{(1-x^4)}}{\cancel{(1-x^2)}}\cdot\frac{\cancel{(1-x^6)}}{(1-x^3)}\cdots\cancel{\cdots}$$

$$\frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}$$

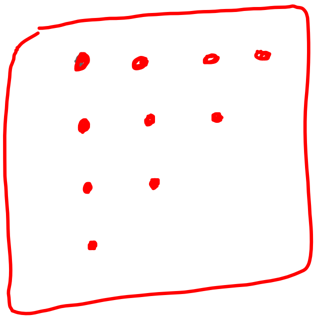
$$\boxed{P_D(x) = P_o(x)}$$

$$\boxed{P_D(n) = P_o(n)}$$

$$\boxed{10 = 6+\underset{\smile}{2}+\underset{\smile}{1}+1}$$

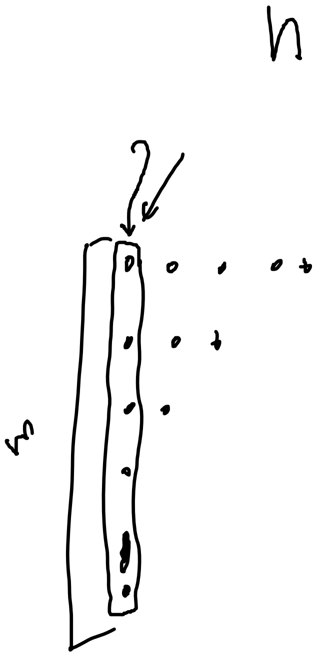


$10 = \underset{\smile}{4}+\underset{\smile}{3}+\underset{\smile}{2}+1$



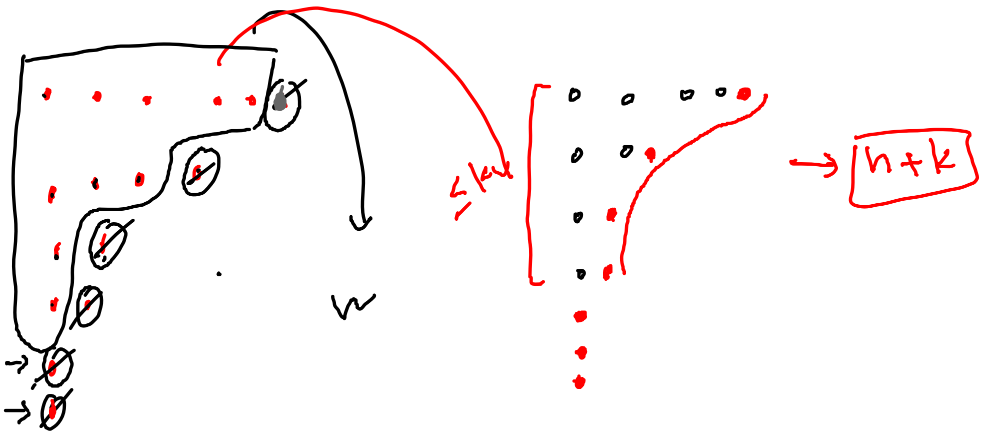
$$\boxed{4+2+1+1+1+1=10}$$

Ferrers



$\boxed{P_m(n)}$ = # partitions of n to summands where the largest summand equals m

$$P_k(n+k) = P_{\le k}(n)$$



$10 = \underbrace{10}$ ←

odd number parts
and distinct
summands

$10 = \overline{9} + \overset{\frown}{1}$

$10 = \boxed{\cancel{7+1+1+1}}$

$\boxed{P_D(n)}$ ✓

↑

$10 = 8 + 2$

$$\left.\begin{aligned} n &= \frac{3m^2 - m}{2} \\ n &= \frac{3m^2 + m}{2} \end{aligned}\right]$$

$n=1$

$P_{DO} = 1$

1

$P_{DE} = 0$

$P_{DO}=1$

$P_{DE}=0$

$n=2$

$\cancel{=1+1}$

$$n=1 = \frac{3m^2-m}{2} = \frac{3-1}{2} = 1$$

$$n = \frac{3m^2+m}{2} = \frac{3+1}{2} = 2$$

$n=3$

$=2+1$

~~$=1+1+1$~~

$P_{DO} = 1$

$P_{DE} = 1$

1

2

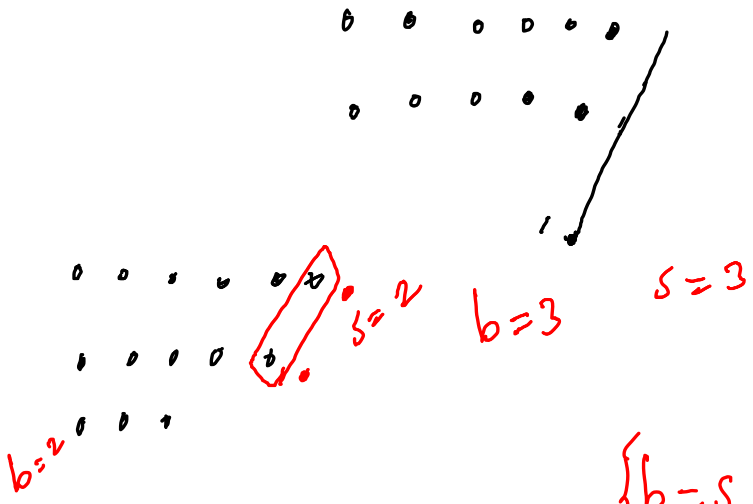
5 ✓

7 ✓

$n=3$

$$\frac{3m^2-m}{2}$$

$$\frac{3m^2+m}{2}$$



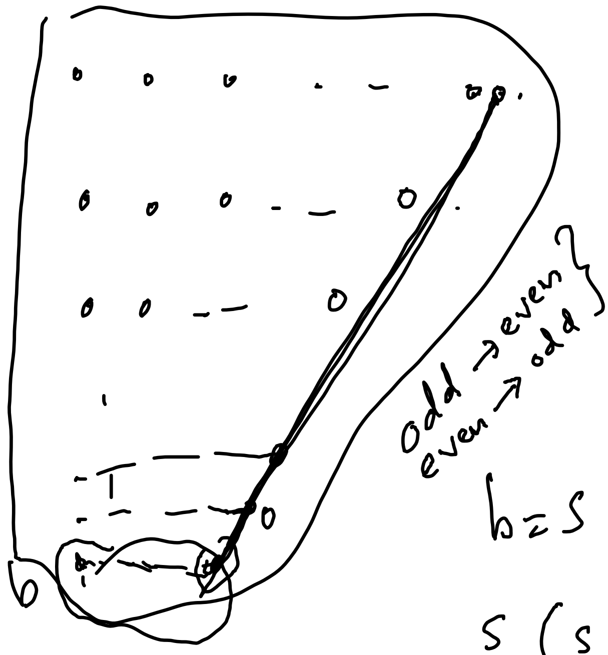
$b \leq s$

$\boxed{b = s}$

$\left\{ \begin{array}{l} b = s \text{ and the slope} \\ \text{share a point with base} \end{array} \right.$

n

$= w(s)$



$b = s$

$s + (s+1)$

$(s+2) +$

$(s+3) + \cdots$

$(2s-1)$

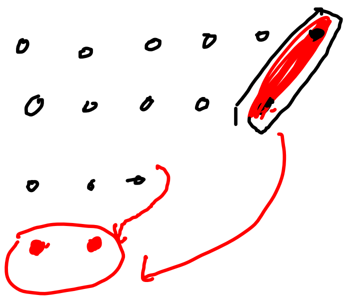
$\frac{s}{2}\left(s + 2s-1 \quad \right)$

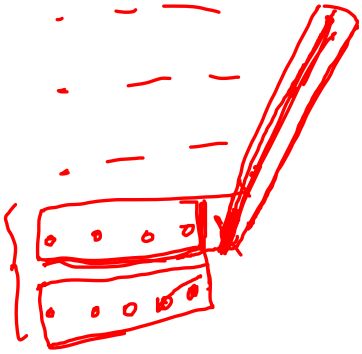
$$\frac{s}{2}(3s-1)$$

$$w(s) = \frac{3s^2-s}{2}$$

$\boxed{b \le s}$ $n \neq w(s)$

case2: if $b > s$ ✓



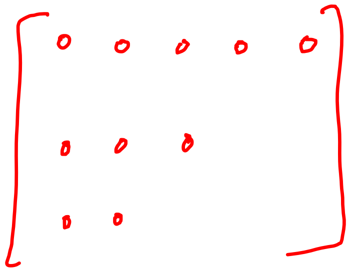


$s < b$

~~$s \nless b-1$~~

$s = b-1$

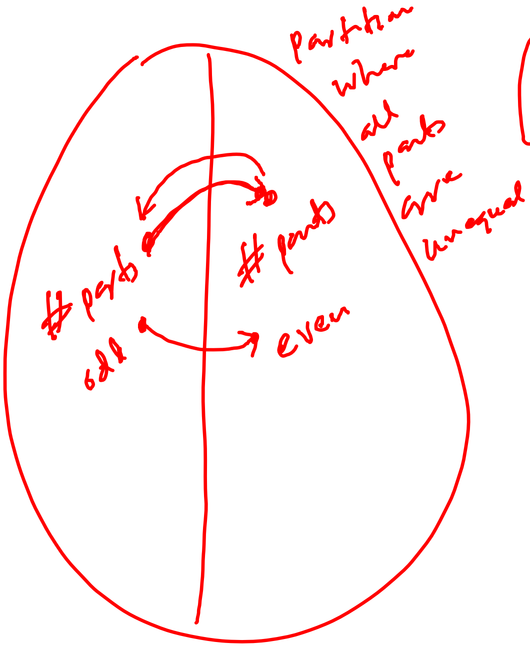
$\underbrace{S=b-1}$

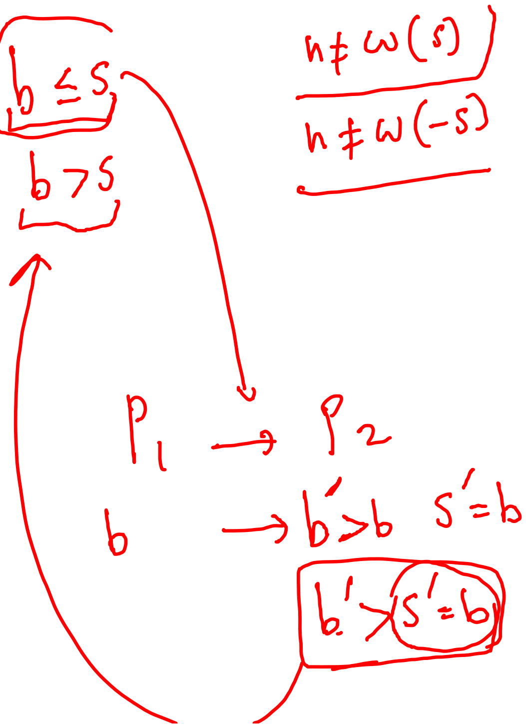


$(S+1)+(S+2)+(S+3)+(S+5)$

$(S+1)+\cdots+(2S)=\frac{S}{2}(3S+1)$

$=\frac{3S^2+S}{2}=\boxed{\omega(-S)}$

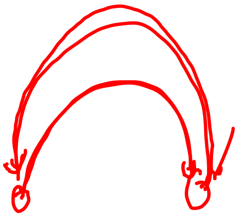




$n \neq \omega(s)$

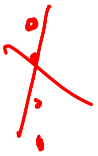
$n \neq \omega(-s)$

$\boxed{n = \omega(-s)}$



$b \leq s$

$b > s$



$s' \geq s = b'$

$\boxed{s < b}$

$b' = s$

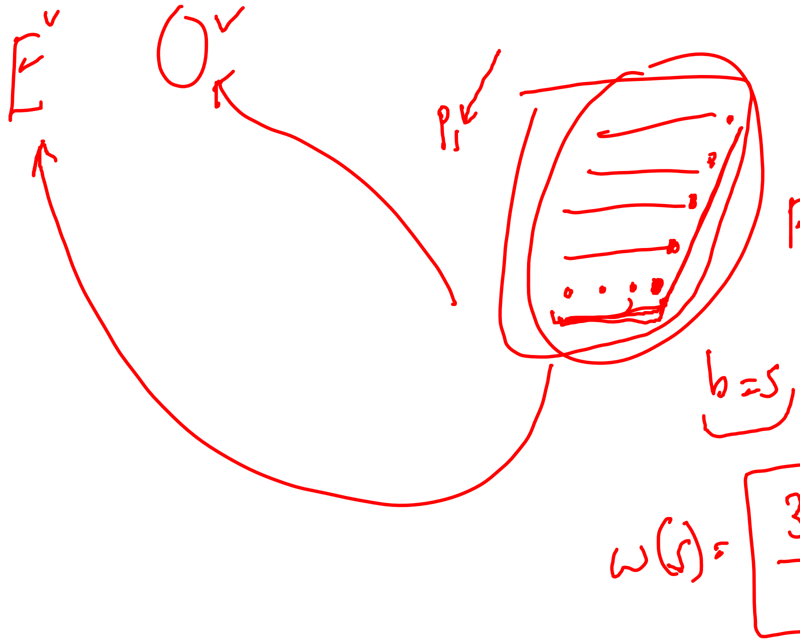
$\boxed{s' \geq b'}$ ✓

$$\begin{bmatrix} n \neq \omega(s) \\ n \neq \omega(-s) \end{bmatrix}$$

$$|O| = |E|$$

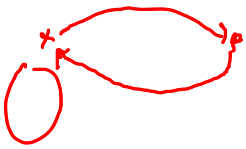
$$O \to E$$

$$n = \omega(s)$$



$$\boxed{n = \omega(s)}$$

$$\omega(s) = \boxed{\frac{3s^2 - s}{2}} ✓$$



$\underbrace{E - \{p_1\}}$

$\boxed{S \text{ even}}$

$|O| = |E| - 1$

$\underline{n = \omega(s)}$

$\neq \omega(-s)$

$\cancel{\omega(-s)}$

$\underbrace{|O|} + 1 = \underline{|E|}$

$n \neq \omega(r)$

$n \neq \omega(-s)$

$\boxed{|O| = |E| + 1}$

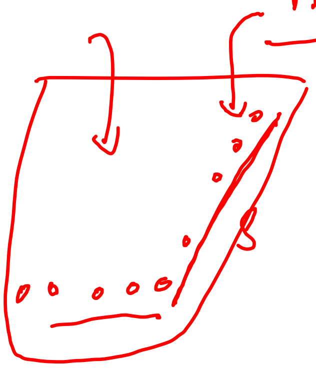
$n = \cancel{\omega}(s)$

$|E - \{p_1\}| = |O|$

$\underline{n = \omega(-s)}$

$|E| = |O| + 1 \checkmark$ p'_1

$|O| = |E| + 1 \checkmark$

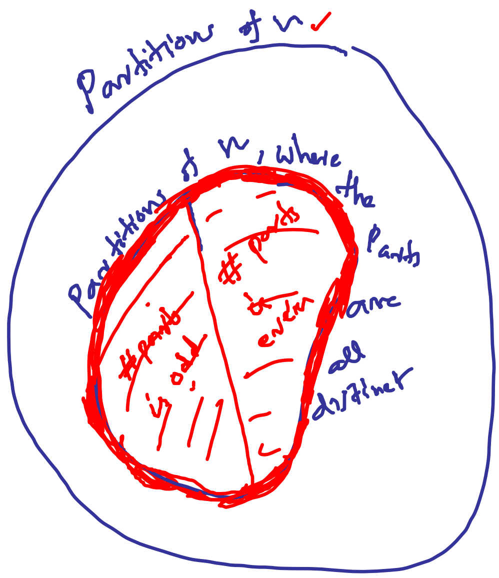


$b > s$

$b = s + 1$

$\frac{3s^2 + s}{2} = \omega(-s)$

n



$$n = \omega(m) = \boxed{\frac{3m^2 - m}{2}}$$

$$= \omega(-m) = \frac{3m^2 + m}{2}$$

$$\omega(m) = \left[\frac{3m^2 - m}{2}\right] \qquad \frac{3m^2 + m}{2} = \omega(-m)$$

$m=1$	1	2
$m=2$	5	7
$m=3$	12	15
	k	k

$m_1 \quad m_2$

$$\boxed{\frac{3m_1^2 - m_1}{\cancel{2}} = \frac{3m_2^2 + m_2}{\cancel{2}}}$$

$$3(m_1^2 - m_2^2) = (m_1 + m_2)$$

$$3\cancel{(m_1 + m_2)}(m_1 - m_2) = \cancel{(m_1 + m_2)}$$

$$\boxed{(m_1 - m_2) = 1/3}$$

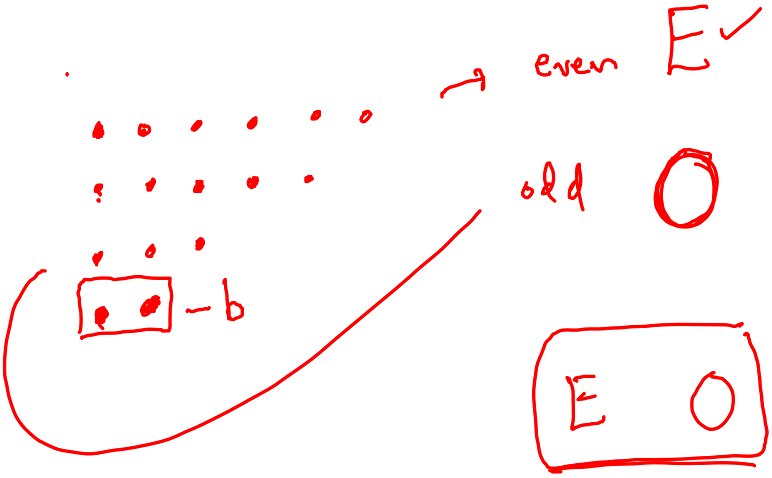
m_1 and m_2

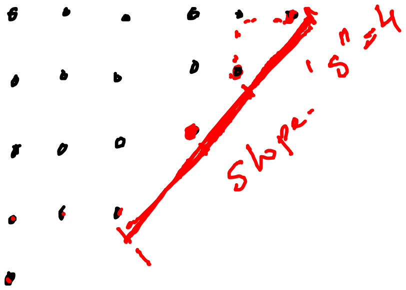
$$\omega(m_1) = \omega(-m_2) \quad \alpha$$

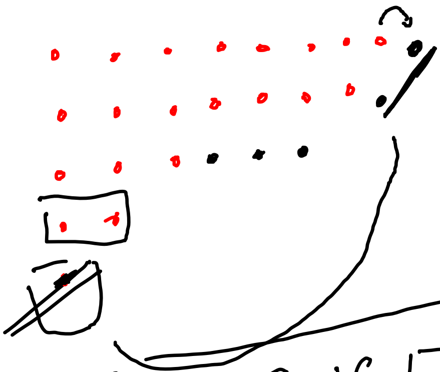
Proof by Ferrer's diagram

n = w(m)

w(-m)

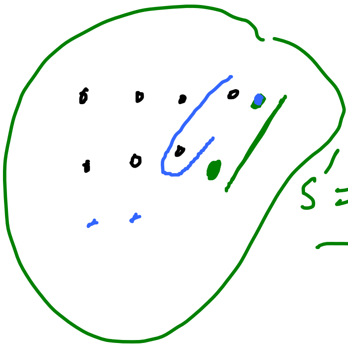






$\boxed{S = b}$

case ① if $\boxed{b \le S}$



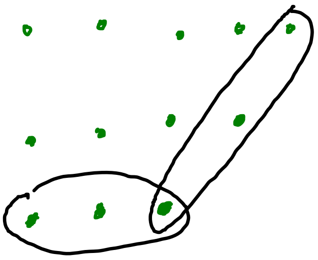
$s' = b = 2$

$b = 2$

$S = 3$

$\boxed{b \le S}$

$b < S$



$b=3$

$E \qquad O$

$|E| \neq |O|$, for all

n where

$n \neq \omega(s)$ ✓

$n \neq \omega(-s)$ ✓

$$\omega(s) = \frac{3s^2 - s}{2} \checkmark$$

$$\omega(-s) = (3s^2+s)/2 \checkmark$$

$$\prod_{i=1}^{\infty}(1 \textcircled{+} x^i)$$

↓

$$\prod_{i=1}^{\infty}(1 - x^i) = \cdots\cdots \quad \boxed{}\, \underbrace{x^n}$$

$$(1-x^1)(1-x^2)(1-x^3)\cdots\cdots(1-x^6)$$

odd

$$x^{\textcircled{n}} = (-x^1)(-x^3)(-x^6)$$

$$\overset{10}{\swarrow}$$

$$= \underbrace{(-1)}\,\underbrace{x^{10}}$$

$$x^{10}$$

$$\underbrace{\overset{\checkmark}{1} + \overset{\checkmark}{3} + \overset{\checkmark}{6}}$$

$\cancel{0x^n}$

$10 \leftarrow \omega(-s)$

$\omega(+s)$

"n"

s

$$\underbrace{(-1)^s x^{\omega(s)}}_{\checkmark} \qquad \underbrace{x^{\omega(-s)}}_{\checkmark}$$

$$\prod_{i=1}^{\infty}(1-x^i) = 1 + \sum_{m=1}^{\infty}(-1)^m\left[x^{\omega(m)} + x^{\omega(-m)}\right]$$

$p(0), p(1), p(2), \ldots$

$$\frac{1}{p(x)}$$

$$P(x) = \prod_{i=1}^{\infty}(1-x^i)^{-1}$$

$$\frac{1}{p(x)}\, p(x) = \underline{1}$$

$$\underbrace{\left[1+\sum_{m=1}^{\infty}(-1)^{m}\left[\underline{x}^{\omega(m)}+x^{\omega(-m)}\right]\right]}\;\overset{p(x)}{\downarrow}\;\left[\underline{p(0)+p(1)x+\dots\dots\dots}\; p(2)x^{2}+\dots\dots\dots\right]$$

$$=1$$

$$(-1)^{m}\; x^{\omega(m)} \qquad x^{[n-\omega(m)]}$$

$$\downarrow$$

$$p(n-\omega(m))$$

$$x^{\omega(-m)} \qquad x^{n-\omega(-m)}$$

$$(-1)^{m}\, p(n-\omega(-m))$$

$$p(n) + \sum_{m=1}^{\infty} (-1)^{m} \left[p(n-\omega(m)) + p(n-\omega(-m)) \right] = 0$$

✓ $$p(n) = \sum_{m=1}^{\infty} (-1)^{m+1} \left[p(n-\omega(m)) + p(n-\omega(-m)) \right]$$

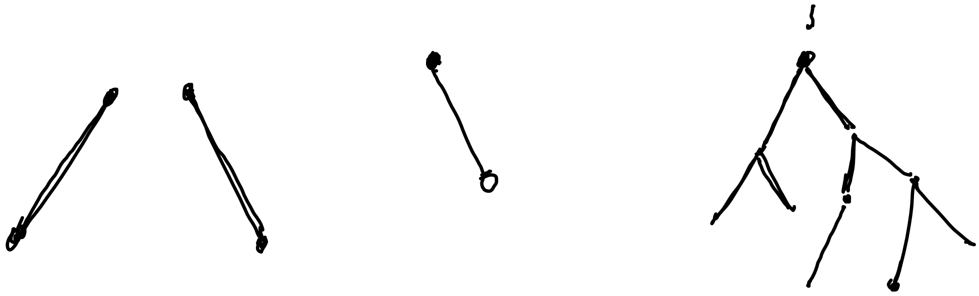
$\boxed{p(k)=0}$ $k<0$

$$\boxed{\frac{3m^2-m}{2} > n}$$

J. H. Van Lint & R. M. Wilson

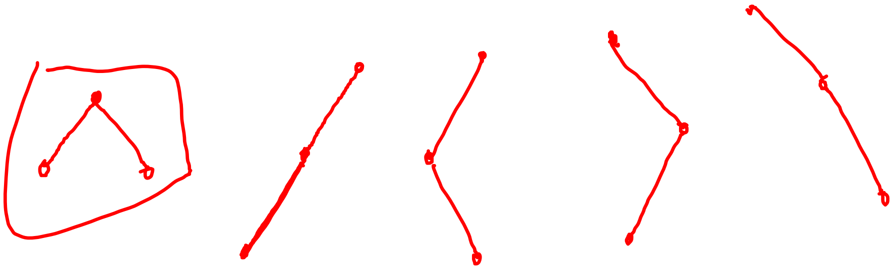
(A course in Combinatorics

chapter 15)



$b_1 = 1$

$b_2 = 2$

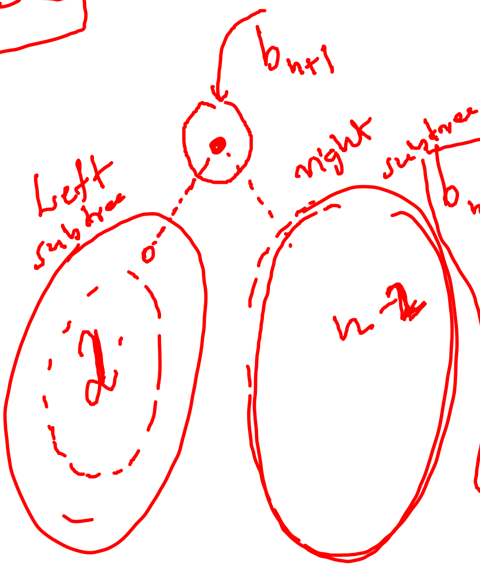


$b_3 = 5$

14?

$\boxed{b_0 = 1}$ ✓

$b_n, b_{n-1}, \dots, b_0$



$$\boxed{\begin{aligned} b_{n+1} &= b_0 \cdot b_n + b_1\, b_{n-1} \\ &\quad b_2 \cdot b_{n-2} + \cdots \\ &\quad + b_n b_0 \end{aligned}}$$

$$b_4 = b_0 b_3 + b_1 b_2 + b_2 b_1 + b_3 \cdot b_0$$

$$\underbrace{1 \cdot 5} + \underbrace{1 \cdot 2} + \underbrace{2 \cdot 1} + \underbrace{5 \cdot 1}$$

$$= \underline{14}$$

$$b_n = ?$$

$$\boxed{B(x) = \sum_{n=0}^{\infty} b_n x^n}$$

$$\sum_{n=0}^{\infty} b_{n+1} x^{n+1} = \sum_{n=0}^{\infty} \left[b_0 b_n + \cdots + b_n b_0 \right] x^{n+1}$$

for $n \geq 0$

$$\left[B(x) - 1 \right] =$$

$$B(x)-1 = x\sum_{n=0}^{\infty}\underline{\left[b_0b_n+b_1b_{n-1}+\cdots+b_nb_0\right]}x^n$$

$$= x\left[B(x)\right]^2$$

$$B(x)-1 = x\left[B(x)\right]^2$$

$$\boxed{B(x)^2} = B(x)\cdot B(x)$$

$$= \left[\underbrace{b_0}_{}+\underbrace{b_1}_{}x+\cdots\cdots\right]\left[b_0+b_1x+b_2x^2+\cdots\right]$$

$$\left[\underline{b_0b_n+b_1b_{n-1}+\cdots b_nb_0}\right]x^n$$

$$\underbrace{x}\,\underbrace{\left[B(x)\right]^2} - \underbrace{B(x)} + 1 = 0$$

$$\boxed{B(x) = \frac{+1 \pm \sqrt{1-4x}}{2x}}$$

$$\sqrt{(1-4x)} = (1-4x)^{1/2}$$

$$= 1 + \binom{1/2}{1}(-4x)^1 + \binom{1/2}{2}(-4x)^2 + \cdots$$

$$\Rightarrow \binom{1/2}{n}(-4)^n x^n$$

$$\binom{\frac{1}{2}}{n}(-4)^n$$

$$\frac{\frac{1}{2}\overset{-1/2}{\left(\frac{1}{2}-1\right)}\overset{-3/2}{\left(\frac{1}{2}-2\right)}\overset{-5/2}{\cdots}\overset{-(2n-3)/2}{\left(\frac{1}{2}-(n-1)\right)}}{n!}(-4)^n$$

$$= \frac{(-1)(-3)\cdots-(2n-3)(-2)^n}{n!}$$

$$\frac{(-1)\boxed{2^n}\left(1\cdot\overset{2\cdot1}{}3\cdot\overset{2\cdot2}{}5\cdot\overset{2\cdot3}{}\overset{2\cdot4}{7}\cdots(2n-3)\overset{(2n-2)}{}\right)}{n!\,(n-1)!}$$

$$\frac{(-1)2\,((2n-2)!)}{n\,(n-1)!\,(n-1)!}$$

$$= \frac{-2}{n}\binom{2n-2}{n-1}$$

$$\sqrt{1-4x}$$

$$\frac{1}{2x}\left[1 \underset{-}{\cancel{+}} \left[1 + \cdots \frac{-2}{n}\binom{2n-2}{n-1}x^n + \cdots\right.\right.$$

$$= \frac{1}{\cancel{2}\cancel{x}}\left[\frac{x}{1}\overset{1}{\cancel{x}} + \cdots x^{\cdot} \cdots + \underbrace{\frac{x}{n}\binom{2n-2}{n-1}}\underline{x^n} + \cdots\right.$$

$$x^{n-1}$$

$$\frac{1}{n}\binom{2n-2}{n-1}x^{n-1}$$

x^n

$$b_n = \boxed{\frac{1}{n+1}\binom{2n}{n}} \leftarrow$$

n th Catalan number

$$\frac{1}{n+1}\binom{2n}{n} \cdot$$

$C_0 = 1$

$C_1 = 1$

$C_2 = 2$

$C_3 = 5$

$\frac{1}{3} \cdot 6 = 2$

$\frac{1}{4}\binom{6}{3}$

$\frac{1}{\cancel{4}} \times \frac{\cancel{6} \cdot 5 \cdot \cancel{4}}{\cancel{3!}}$

$\overset{+1}{\textcircled{a_1}}\quad \overset{\pm 1}{a_2}\ \cdots\cdots\ \overset{\pm 1}{\underline{a_{2n}}}$

$\boxed{b_0 = 1}$

$\boxed{b_1 = 1}$

$\overset{1}{a_1}\ \overset{-1}{a_2}$

$\underbrace{\overset{1}{a_1}\ \overset{-1}{a_2}}\ \underbrace{\overset{+1}{a_3}\ \overset{-1}{a_4}}$

$n=1 \Rightarrow \underline{2n=2}$

$b_2 = 2$

$$\left[\begin{array}{l} \overset{1}{a_1}\ \overset{\checkmark}{\underline{1}}\ \underline{1}\ -1\ -1\ -1 \quad b_3 \quad 1\ -1 \cdots \frac{\textcircled{2}}{\textcircled{$b_{n}2$}} \\ a_1\ a_2\ a_3\ a_4\ a_5\ a_6 \\ 1\ 1\ -1\ 1\ -1\ -1 \\ 1\ 1\ -1\ -1\ 1\ -1 \end{array}\right.$$

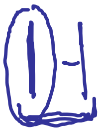
$\underline{\underline{3+2=5}}$

$b_4 = 14\ ?$

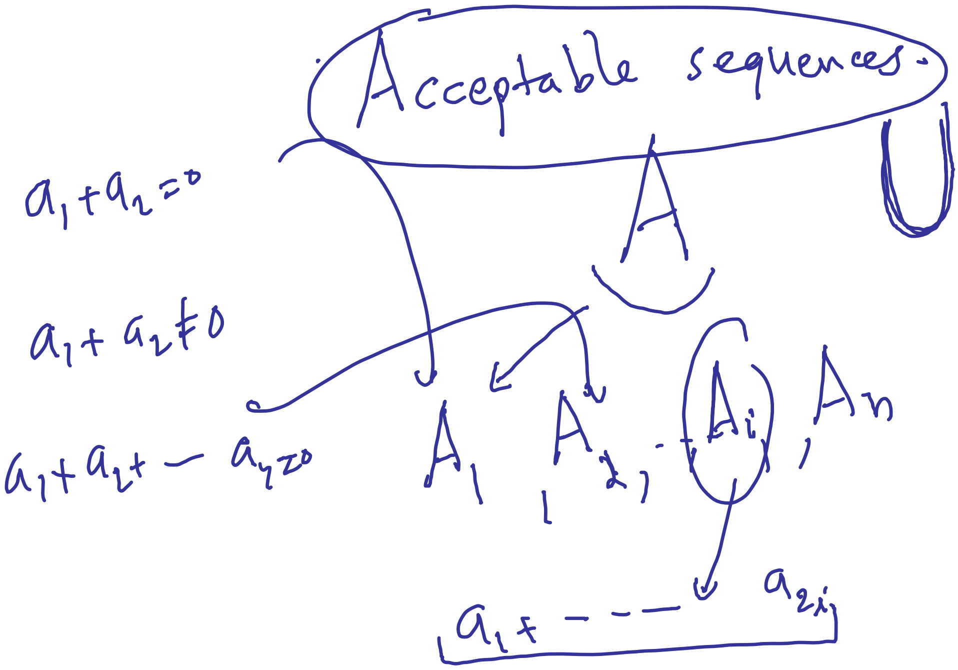
$\underline{a_1+a_2}$

$a_1+a_2+a_3+a_4$

$$\frac{1}{n+1}\binom{2n}{n}$$

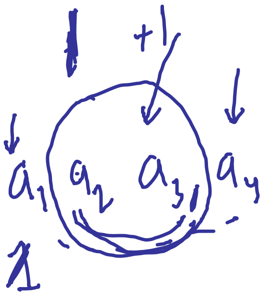


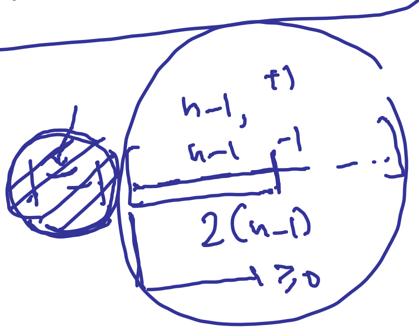
$$\boxed{a_1+a_2+\cdots+\underbrace{a_{2k}}=0}$$



$$|A| = \underbrace{|A_1|} + |A_2| + \cdots + |A_n|$$

$$\boxed{b_n = b_0 b_{n-1} + b_1 b_{n-2} + \cdots + b_{n-1} b_0}$$





✓ $$\boxed{b_{n+1} = b_0 b_n + b_1 b_{n-1} + \cdots + b_n b_0}$$

$$\boxed{\frac{1}{n+1}\binom{2n}{n}}$$ ✓

T $\quad \underset{n}{⊕1} \quad \underset{n}{(-1)}$ $\quad (A) \; (U)$

$$|T| = \binom{2n}{n}$$

$$\frac{2n!}{n!\,n!}$$

U

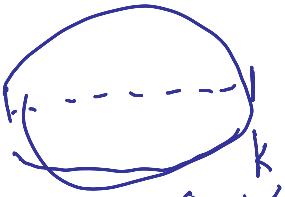
$$\boxed{|A| = |T| - |U| = \frac{1}{n+1}\binom{2n}{n}}$$

$a_1 \; a_2 \; --- \; a_n$

smallest k

$+1$

-1



$a_1' = -a_1$ $\quad a_2' = -a_2$ $\quad a_{k-1}' = -a_{k-1}$

$$\underbrace{a_1 + a_2 + \cdots + a_{k-1}}_{} = 0$$

$a_k' = -a_k$

$+ a_k$ $\quad -1$

$a_{k+1}' = a_{k+1} \cdots a_{2n}' = a_{2n}$

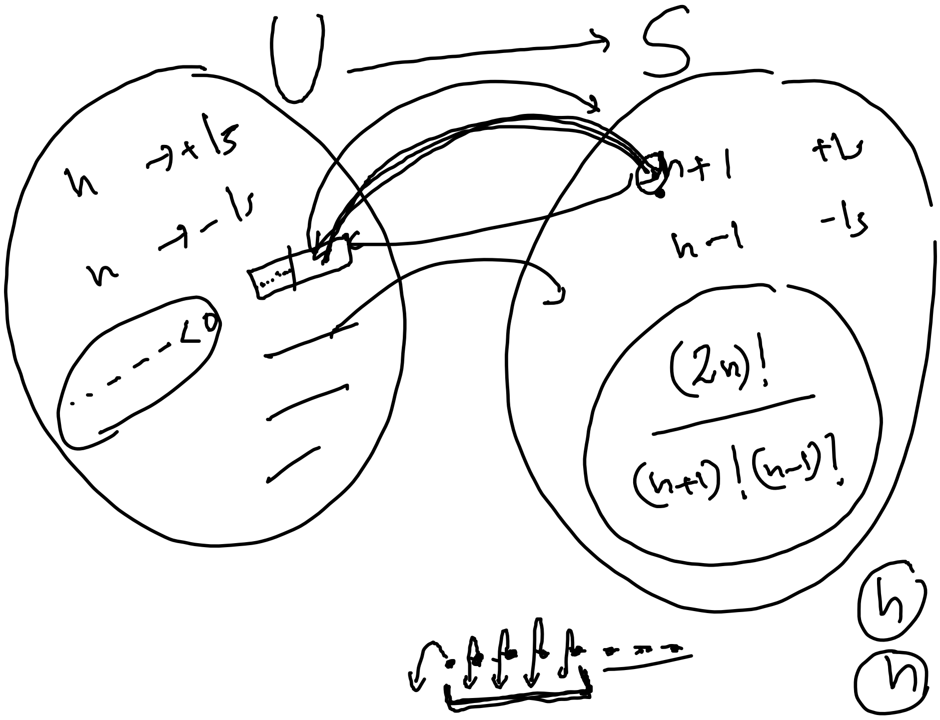
having equal +1 s and -1s

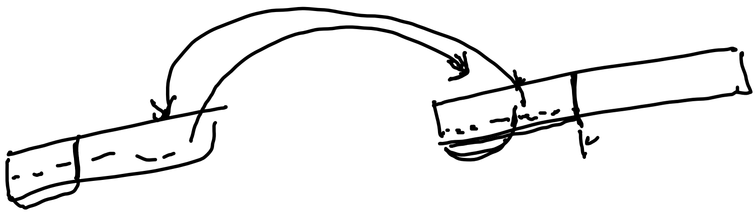
n +1s

n -1s

← n+1 +1s

n-1 -1s





$$\boxed{|U|} = |S| = \frac{2n!}{(n-1)!\,(n+1)!}$$

$$|A| = |T| - |U|$$

$$= \frac{2n!}{n!\,n!} - \frac{2n!}{(n+1)!\,(n-1)!}$$

$$= \frac{2n!}{n!\,(n-1)!}\left[\frac{1}{n} - \frac{1}{n+1}\right]$$

$$= \frac{2n!}{n!\,(n-1)!}\left[\frac{1}{n\,(n+1)}\right]$$

$$= \left[\frac{2n!}{n!\;n!}\right]\frac{1}{n+1}$$

$$= \frac{1}{n+1}\binom{2n}{n} = C_n$$

A B

50 50

n $\boxed{50\text{ RS}}$. — — —

n $\boxed{100\text{ RS}}$

$\underbrace{\cdot\ \ \cdot\ \ \cdot\ \cdot\ \ \cdot}\ \ \cdot\ \ \cdot$

$+1 \quad -1$

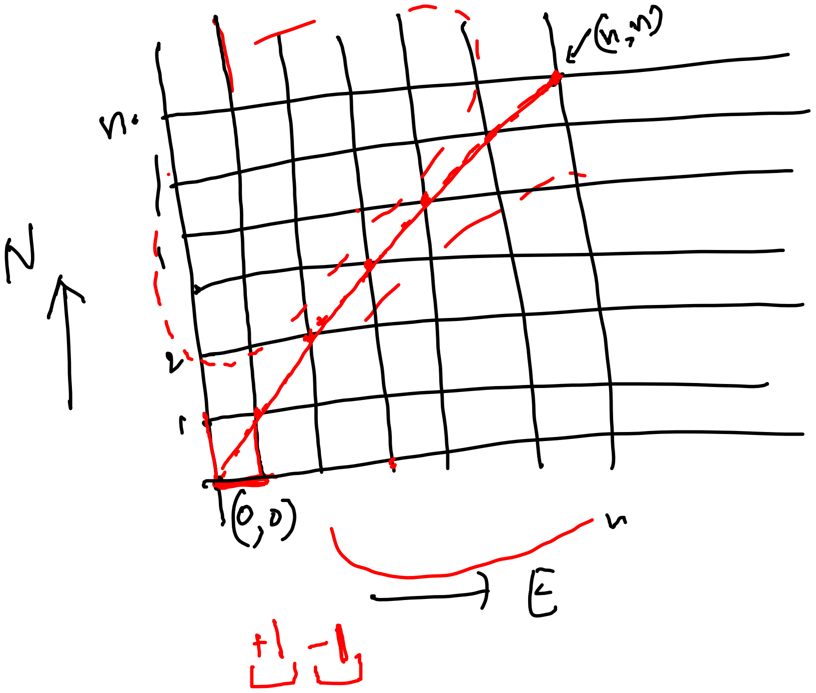
$$\boxed{C_n = \frac{1}{n+1}\binom{2n}{n}}$$

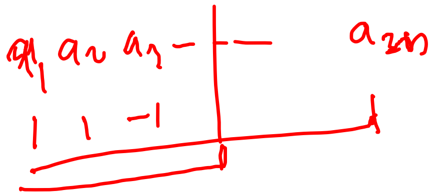
$(n!)(n!)$

$(n!)(n!)\ C_n$

$$\cancel{(n!)^2}\ \frac{1}{n+1}\ \frac{(2n)!}{\cancel{(n!)^2}}$$

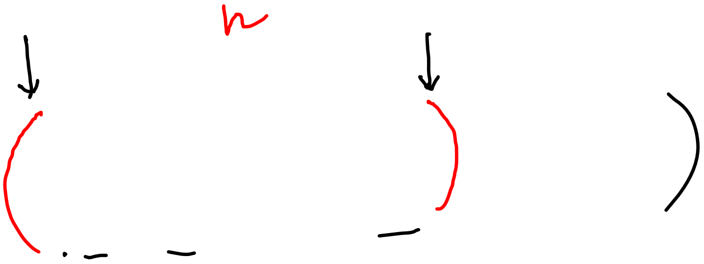
$$= \frac{(2n)!}{n+1} \quad //$$





$$\boxed{a_1 + a_2 + a_3 + \cdots + a_k \geq 0}$$

$$\boxed{\frac{1}{n+1}\binom{2n}{n}}$$



$b_0 = 1$

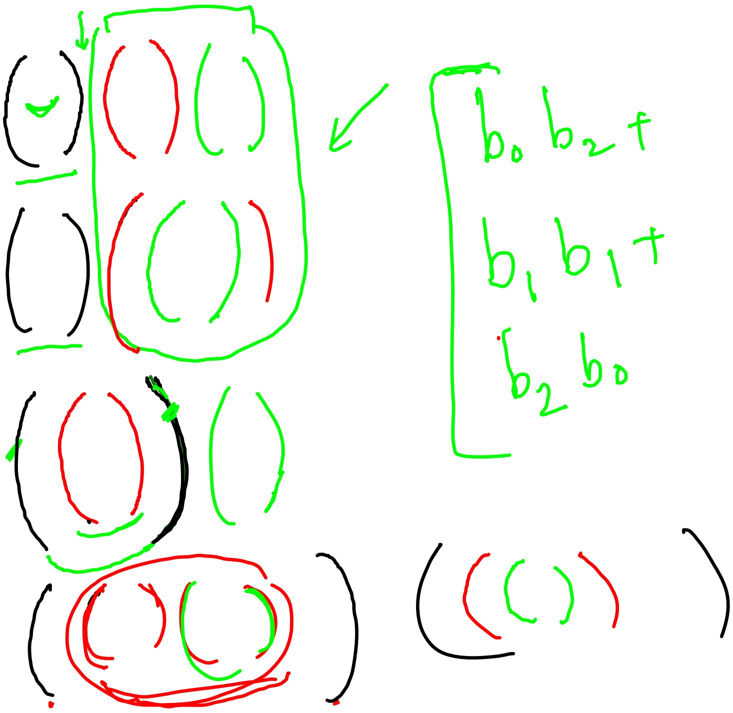
$b_1 = 1$

( )

( ) ( )

(( ))

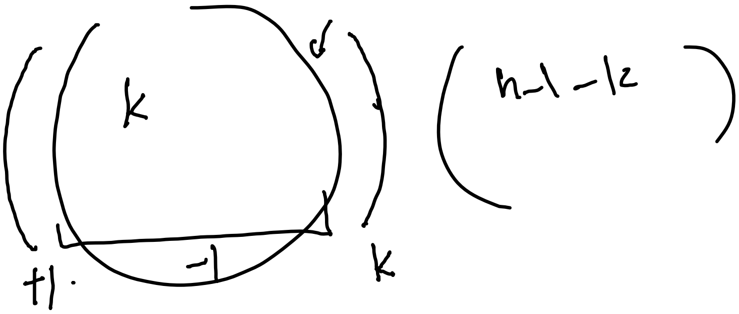
$b_2 = 2$



$b_0 b_2 +$

$b_1 b_1 +$

$b_2 b_0$



$$\boxed{b_n = \sum_{k=0}^{n-1} b_k \cdot b_{n-1-k}}$$

$$b_n = \frac{1}{n+1}\binom{2n}{n}$$

$$\underline{a_1 \cdot ((a_2 \cdot a_3)) \cdot (a_6 \cdot a_7)\, a_{n+1}}$$

$$a_1\ (a' \cdot \cdot) \quad —$$

$2'$

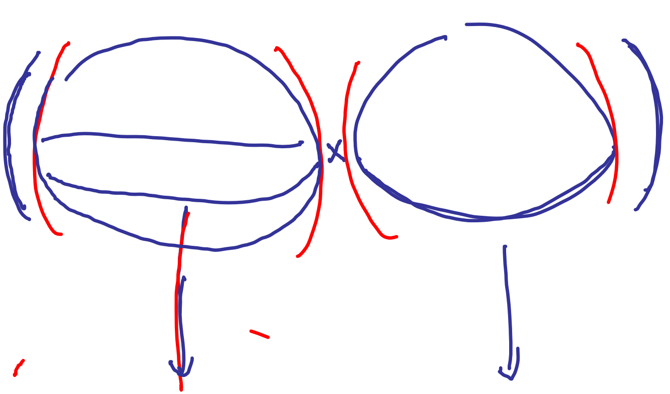
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$a_1\ a_2\ a_3\ a_4 \cdots\ a_{n+1}$ $\quad n+1$

↓

$$\underline{((a_1' \cdot a_2)\cdot(a_3\ (a_4' \cdot a_5)))} \quad n$$

2



$\underline{a_1, \cdots a_{n+1}}$

T_{n+1}

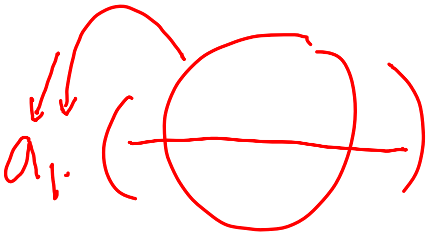
$$\underbrace{a_1 \cdots a_k}_{k\ \text{number}} \cdot \underbrace{a_{k+1} \cdots a_{n+1}}_{(n+1-k)}$$

$$T_k \cdot T_{n+1-k}$$

$$T_{n+1} = \sum_{k=1}^{n} \underbrace{T_k} \cdot \underbrace{T_{n+1-k}}$$

$$T_{n+1} = T_1 \cdot T_n + T_2 T_{n-1} + \cdots + T_n T_1$$

$$\boxed{C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0}$$



$T_1 = C_0 = 1$

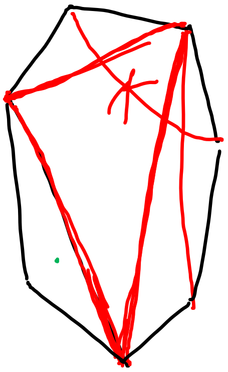
$T_2 = C_1 = 1$

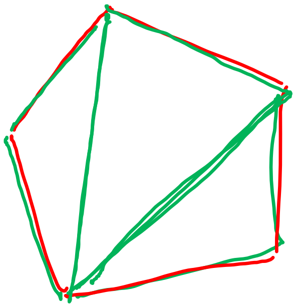
$T_3 = C_2 = 2$

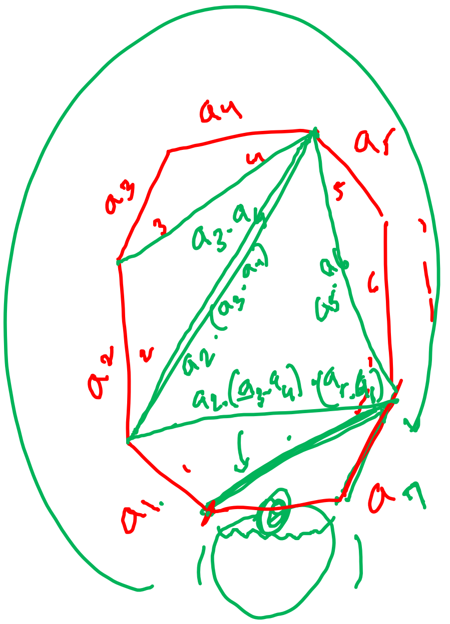
$((a_1 \cdot a_2) \cdot a_3)$

$(a_1 \cdot (a_2 \cdot a_3)$ '

$\underline{T_{n+1} = C_n}$





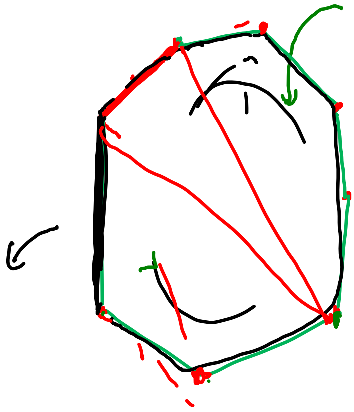


n+2 sides

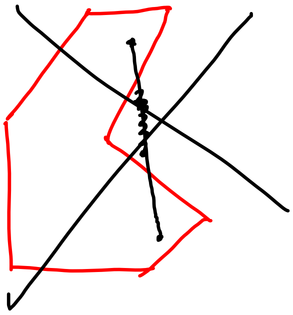
n+1 numbe

$T_{n+2} \to C_n$

$$\boxed{T_{n+2} = C_n}$$



Convex



$T_2 = 1$

$\boxed{T_3 = 1}$

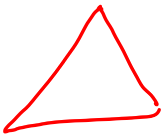
$T_{n+2} = T_2 \cdot T_{n+1} +$

$T_3 \cdot T_n + T_4 T_{n-1} + \cdots + T_{n+1} \cdot T_2$

$$T_{n+2} = C_n$$

$$\boxed{C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0}$$

$$\left.\begin{array}{l} T_2 = C_0 = 1 \\ \hline T_3 = C_1 = 1 \end{array}\right\}$$



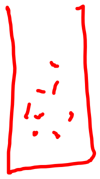
$$\underline{T_n = C_n = \frac{1}{n+1}\binom{2n}{n}.}$$

[The sterling numbers of the 2nd kind]

$$S(p,0)=1 \quad p=0$$

$$\underline{S(p,0)=0 \quad p>0}$$

$S(p,1) = 1$



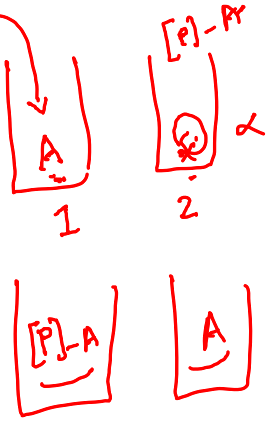
$S(p,p) = 1$

$S(p,k)$

$\boxed{p \geq k}$

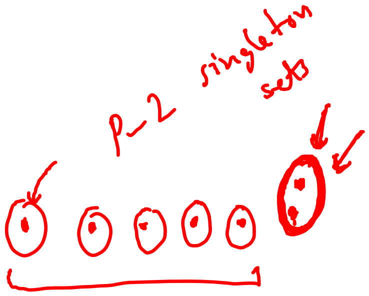
$p \geq 2$ $\boxed{S(p,2) = 2^{p-1} - 1}$

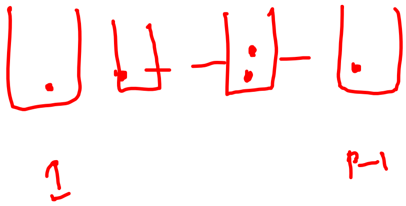
$A \subseteq [p]$



$$\frac{2^p - 2}{2} = \underline{\underline{2^{p-1} - 1}}$$

$$S(p, p-1) = \binom{p}{2}$$





$$\binom{p}{2}$$

$$S(p,k) \leftarrow$$

$$\underbrace{1, 2, --, p}$$

$$\boxed{\begin{array}{l} S(p,p) = 1 \\ S(p,0) = 0 \quad p > 0 \\ \qquad\quad\ = 1, \ p = 0 \end{array}}$$

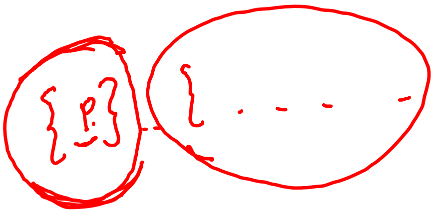
$$\left\{ S(p,k) = k\,S(p-1,k) + S(p-1,k-1) \right\}$$

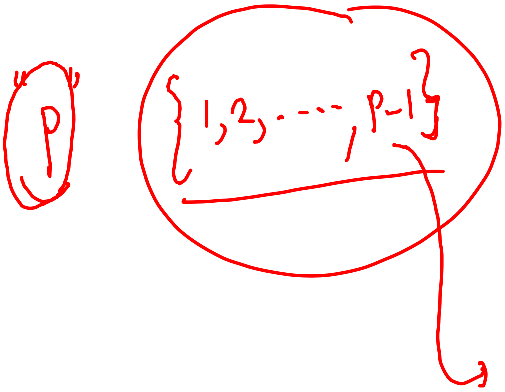
$$1 \le k \le p-1$$

$$\boxed{S(p,p) = 1}$$

$$\boxed{S(p,0) = \begin{matrix} 1 & p>0 \\ 0, & p=0 \end{matrix}}$$

$$S(p,k)$$





$$S(p-1,k-1)$$

$$+$$

$$k \cdot S(p-1,k)$$

$$\rightarrow \boxed{S(p,k) = \underbrace{S(p-1,k-1)} + k\,\underbrace{S(p-1,k)}}$$

$$\underbrace{h_0, h_1}, \underbrace{h_1, h_2}, \underbrace{h_2, h_3}, \underbrace{h_3, ----}$$

first order difference sequence

$$\Delta h_0, \underbrace{\Delta h_1}, \underbrace{\Delta h_2}, \underbrace{\Delta h_3}, ---$$

$$\Delta h_0 = h_1 - h_0$$

$$\Delta h_1 = h_2 - h_1$$

$h_n = \underbrace{2n^2+3n+1}$

0 $\quad h_0 \quad h_1 \quad h_2 \quad h_3 \quad h_4 \quad \dots$

1 $\quad$ 6 $\quad$ 15 $\quad$ 28 $\quad$ 45

$\Delta h_0 \quad \Delta h_1 \quad \Delta h_2 \quad \Delta h_3 \quad \dots$

1 $\quad$ 5 $\quad$ 9 $\quad$ 13 $\quad$ 17 $\quad \dots$

$\Delta^2 h_0 \quad \Delta^2 h_1 \quad \Delta^2 h_2$

4 $\quad$ 4 $\quad$ 4 $\quad \dots$

0 $\quad$ 0 $\quad$ 0 $\quad \dots$

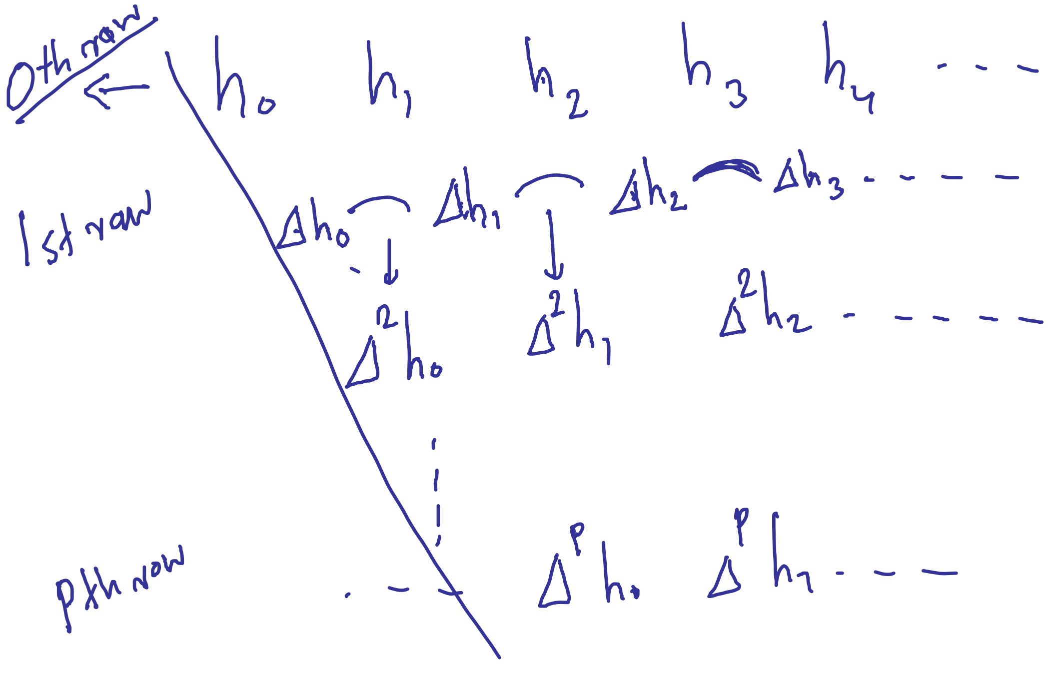
$$\Delta^p h_0 = \Delta\left(\Delta^{p-1} h_0\right) = \Delta^{p-1} h_1 - \Delta^{p-1} h_0$$

$h_0 \quad h_1 \quad h_2 \quad h_3 \quad h_4$

$\Delta h_0 \quad \Delta h_1 \quad \Delta h_2 \quad \Delta h_3$

$\Delta^2 h_0 \quad \Delta^2 h_1 \quad \Delta^2 h_2$

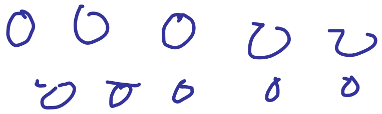
$\Delta^3 h_0 \quad \Delta^3 h_1 \quad \dots$


0th row
h_0 h_1 h_2 h_3 h_4 - - -
1st row
Δh_0 Δh_1 Δh_2 Δh_3 - - - -
$\Delta^2 h_0$ $\Delta^2 h_1$ $\Delta^2 h_2$ - - - -
pth row
$\Delta^p h_0$ $\Delta^p h_1$ - - -

$0 \to h_0, h_1, h_2 \cdots\cdots$

$$\boxed{h_n = a_p n^p + a_{p-1} n^{p-1} + \cdots + a_0}$$

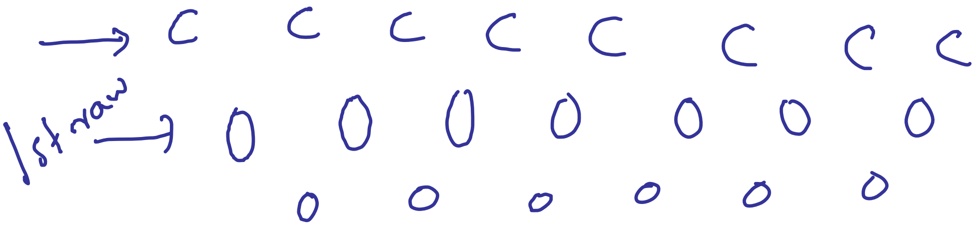
p+1 th row



$$h_n = 2n^2 + 3n + 1$$ ✓

$$\boxed{p = 0}$$

P+1 = 1st raw



$$\boxed{h_n = a_p n^p + \cdots\cdots + a_0}$$

$$\Delta h_n = h_{n+1} - h_n$$

$$= \left[a_p (n+1)^p + \cdots a_0\right] - \left[a_p n^p + \cdots a_0\right]$$

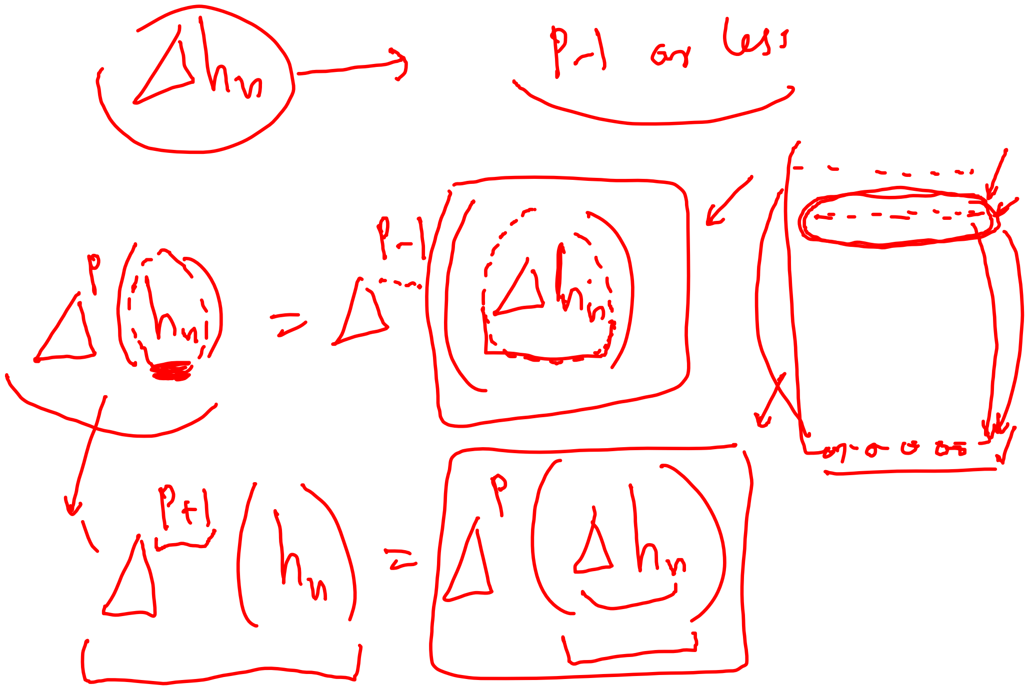
$$n^2$$

$$\downarrow$$

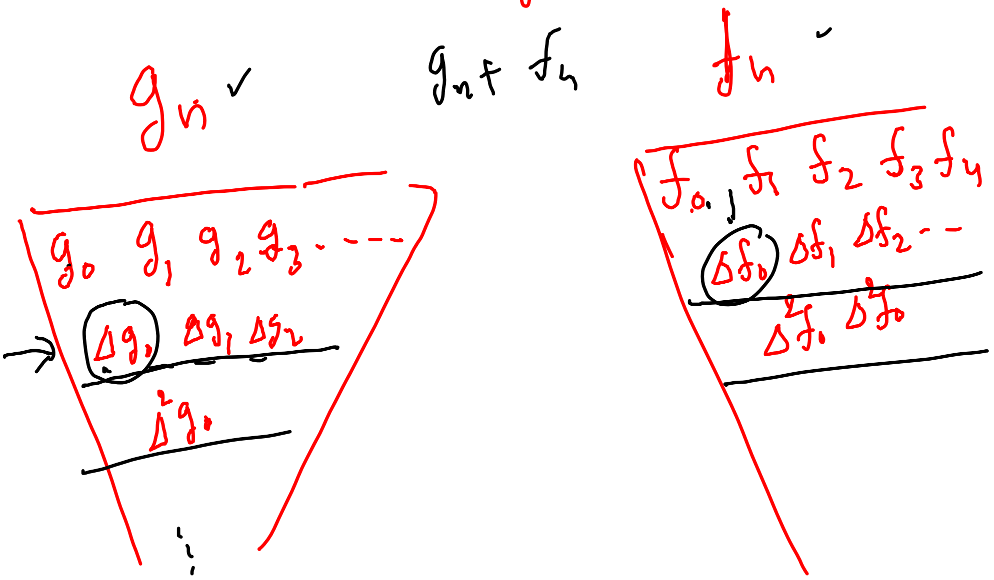
$$a_p (n+1)^p - a_p n^p$$

$$a_p \left[n^p + \binom{p}{1} n^{p-1} \cdots + 1\right] - a_p n^p$$

$$= \boxed{\binom{p}{1} n^{p-1} + \cdots + 1}$$


Δh_n
p-1 or less
Δ^p (h_n) = Δ^{p-1} (Δh_n)
Δ^{p+1} (h_n) = Δ^p (Δh_n)

Linearity


g_n ✓
g_n+f_n
f_n
g_0 g_1 g_2 g_3 - - - -
Δg_0 Δg_1 Δg_2
Δ^2 g_0
f_0, f_1 f_2 f_3 f_4
Δf_0 Δf_1 Δf_2 - -
Δ^2 f_0 Δ^2 f_0

$f_n + g_n$

$g_0 + f_0 \quad g_1 + f_1 \quad g_2 + f_2 \ldots$

$\Delta g_0 + \Delta f_0 \quad \Delta g_1 + \Delta f_1 \quad -$

$\Delta(f_n + g_n)$

$= \Delta f_n + \Delta g_n$

$$\left(\begin{array}{l} c g_0 \quad c g_1 \quad c \cdot g_2 \quad c g_3 \quad c g_4 \ldots \\ \quad c\Delta g_0 \quad c\Delta g_1 \quad c\Delta g_2 \ldots - \\ \qquad c\Delta^2 g_0 \quad c\Delta^2 g_1 \ldots \\ \qquad\quad c\Delta^3 g_0 \ldots \end{array} \right.$$

$g_n \qquad\qquad c\, g_n$

$$\Delta(c\, g_n) = c\, \Delta g_n \checkmark$$

$$\boxed{\Delta^p(c\, g_n) = c\, \Delta^p(g_n)}$$

$$\Delta^p(c\, g_n) = \boxed{\Delta}\left(\boxed{\Delta^{p-1}(c\, g_n)}\right)$$

$$= \Delta\left(c\, \Delta^{p-1}(g_n)\right)$$

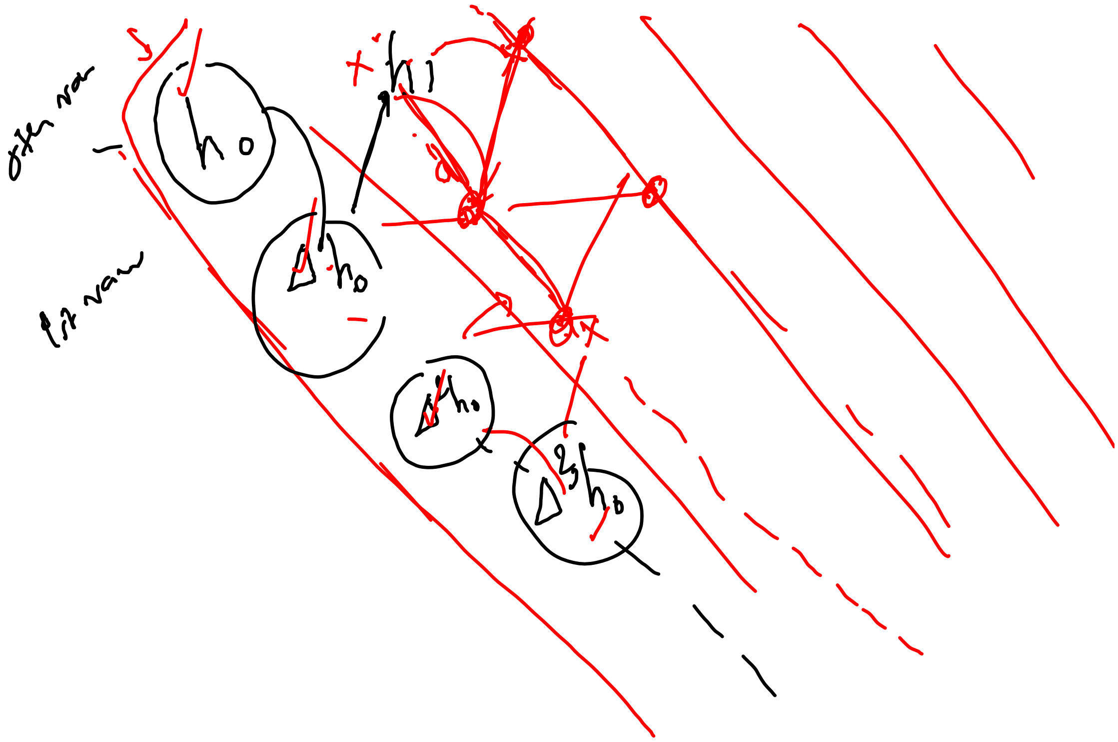
$$= c\, \Delta\left(\Delta^{p-1}(g_n)\right)$$

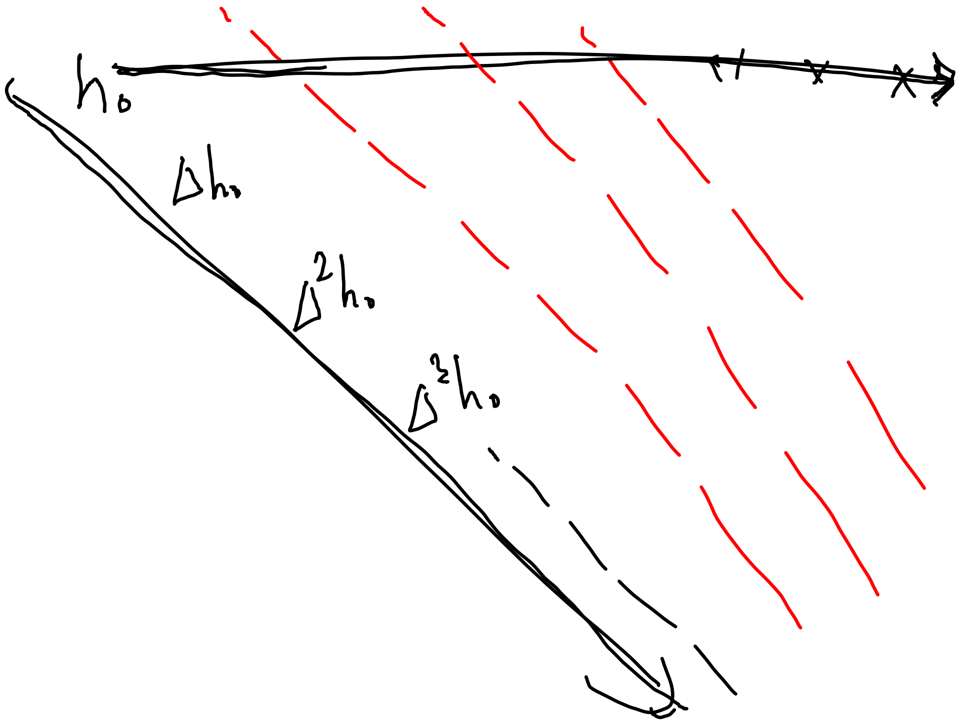
$$= \underline{c\, \Delta^p(g_n)}$$

$$\Delta^p(g_n + f_n) = \Delta^p(g_n) + \Delta^p(f_n)$$

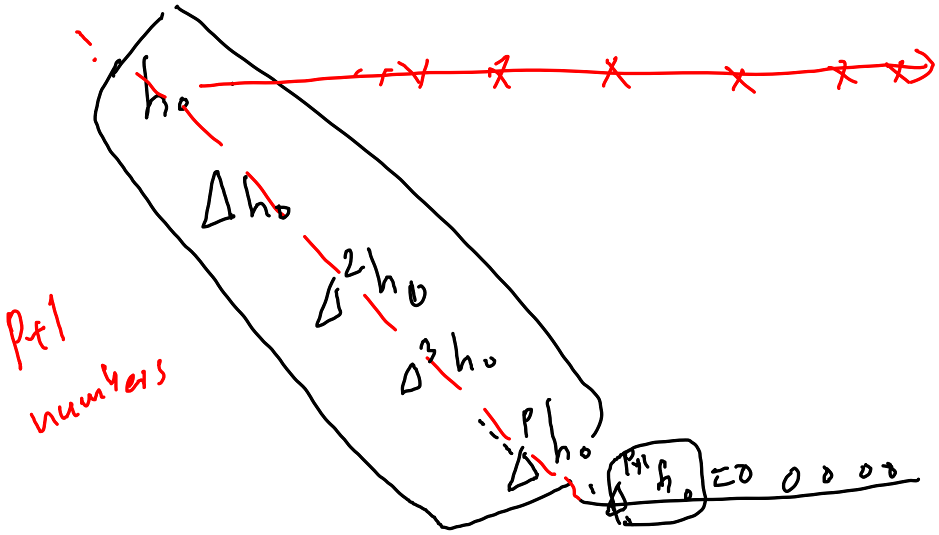
$$\Delta^p\left(c\, g_n + d\, f_n\right) = \Delta^p(c\, g_n) + \Delta^p(d f_n)$$

$$= c\,\Delta^p(g_n) + d\,\Delta^p(f_n)$$


0th row
1st row
h₀
Δh₀
Δ²h₀
Δ³h₀
h₁


h₀
Δh₀
Δ²h₀
Δ³h₀

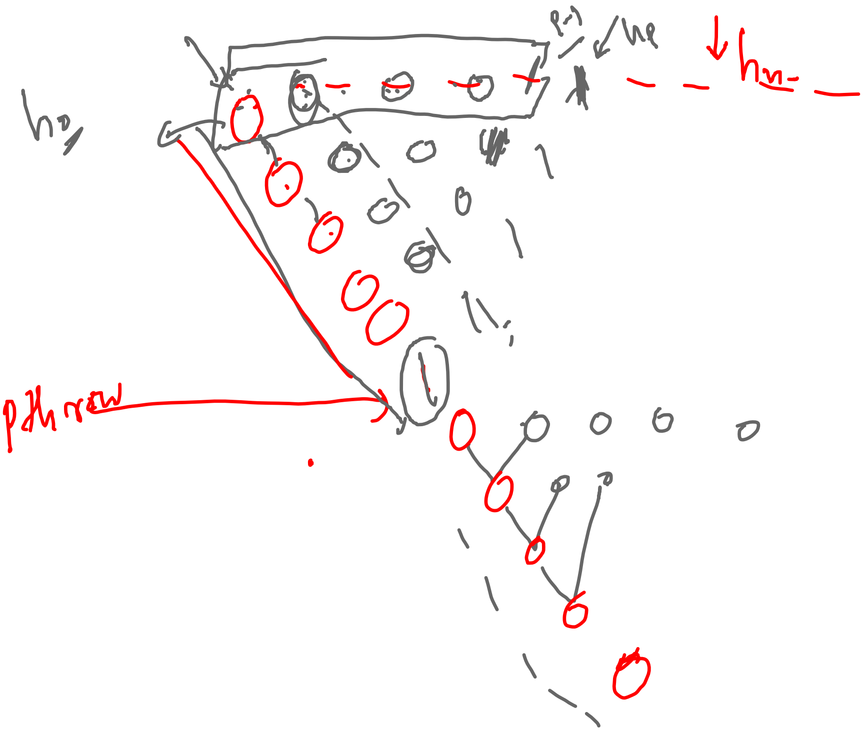
$$h_n = \underbrace{a_p n^p + a_{p-1} n^{p-1} + \cdots + a_0}$$



$$\boxed{h_n = c_p n^p + \cdots + c_0}$$

$$\underbrace{f_n}_{} = \boxed{c_0 \underbrace{e_n^0}_{}} + \underbrace{c_1}_{} e_n^1 + \cdots + \underbrace{c_p}_{} e_n^p$$

1	0	0
0	1	0
0	0	1
1	0	0
1	0	0



$h_0 = 0$ $\quad H(n)$

$= c\,(n-0)(n-1)\cdots(n-p+1)$

$h_1 = 0$

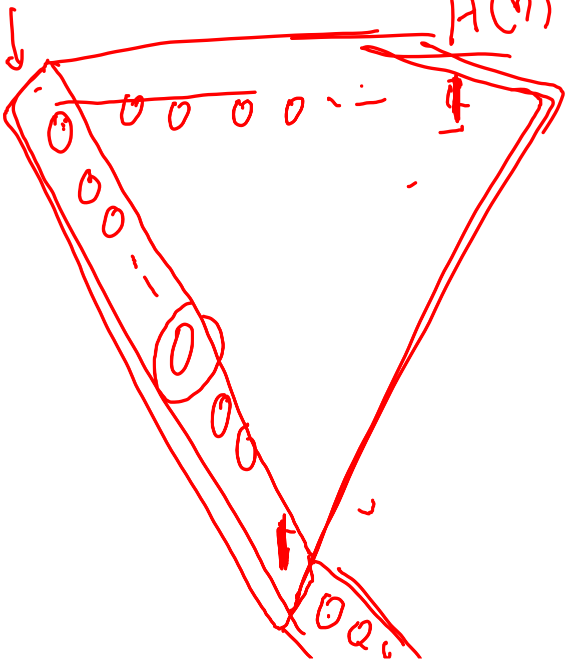
$\underline{H(n) = c\;n(n-1)(n-2)\cdots(n-p+1)}$

$\vdots$

$h_{p-1} = 0$ $\quad H(p) = c\;\underline{p(n-1)\cdots 1}$

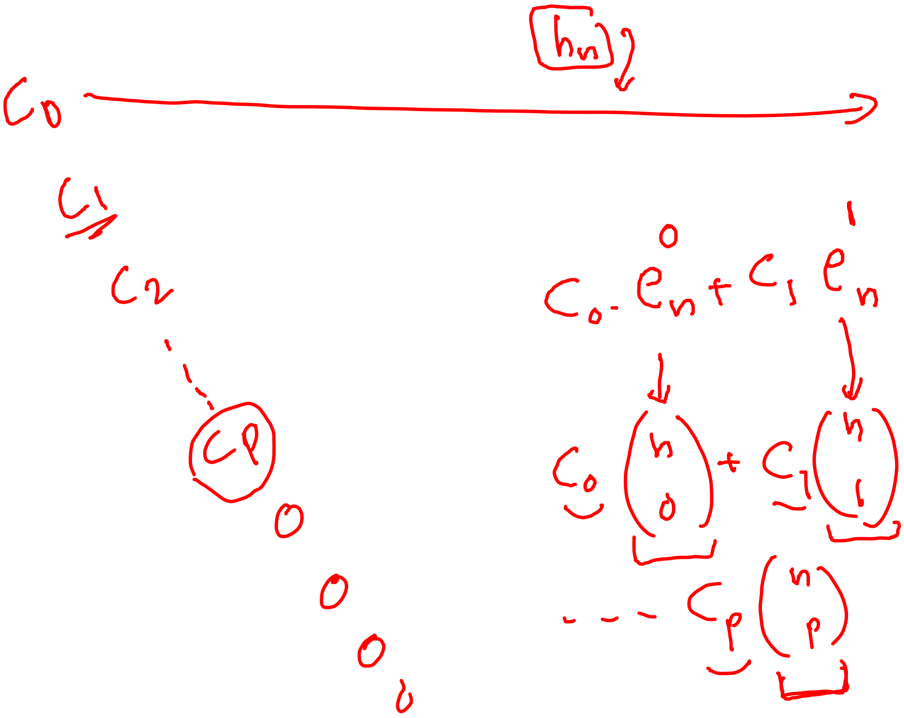
$\boxed{h_p = 1}$ $\quad h_p = 1 = c\,p!$

$c = 1/p!$



$$H(n) = \frac{1}{p!}\left[n(n-1)\cdots(n-p+1)\right]$$

$$= \underbrace{\binom{n}{p}}_{} = \boxed{\frac{n^{p}}{p!}}$$



$$C_0 \cdot e_n^{0} + C_1 e_n^{1}$$

$$\downarrow \qquad \downarrow$$

$$C_0 \underbrace{\binom{n}{0}}_{} + C_1 \underbrace{\binom{n}{1}}_{} + \cdots C_p \underbrace{\binom{n}{p}}_{}$$

c_0

c_1

c_2

$\vdots$

c_p

0

0

0

0

$$\boxed{h_n = \sum_{i=0}^{p} c_i \binom{n}{i}}$$

$$H(n) = \sum_{i=0}^{p} c_i \frac{n^i}{i!}$$

$$= \underbrace{\sum_{i=0}^{p} \left(\frac{c_i}{i!}\right) n^i}$$

c_0

c_1

c_2

$\vdots$

c_p

0

0

$H(n)$

$$\boxed{n^{\underline{p}} = n(n-1)(n-2)\cdots(n-p+1)}$$

$$1,\ x,\ x^2,\ \ldots,\ x^p$$

$$1,\ n,\ n^2,\ \ldots\ n^p$$

$$\binom{n}{0},\ \binom{n}{1},\ \binom{n}{2}\ \ldots,\ \binom{n}{p}$$

$$n^{\underline{0}}\quad n^{\underline{1}}\quad n^{\underline{2}}\ \ldots\ n^{\underline{p}}$$

$$n^p = \sum_{i=0}^{p} \boxed{C(p,i)} \binom{n}{p}$$

$$n^p = C(p,0)\binom{n}{0} + C(p,1)\binom{n}{1} + \cdots + C(p,p)\binom{n}{p}$$

$$= \frac{C(p,0)}{0!} n^0 + \frac{C(p,1)}{1!} n^1 + \cdots + \frac{C(p,p)}{p!} n^p$$

$$\boxed{\frac{C(p,k)}{k!}} \leftarrow \boxed{S(p,k)}$$

Second kind

$$\boxed{S(p,k)}$$

$$\underbrace{\{1, 2, \ldots, p\}}$$

"k"

non-empty

subsets

$$n^{\underline{p}}$$

$$\underline{n}^{p} = \frac{C(p,0)}{0!}\, n^{\underline{0}} + n^{\underline{1}} \quad n^{\underline{2}} \quad \ldots \; n^{\underline{k}} \quad n^{\underline{p}}$$

$$n^{\underline{1}} \downarrow \frac{C(p,1)}{1!} \qquad n^{\underline{k}} \downarrow \frac{C(p,k)}{k!} \; \cdots$$

$$S'(p,k) \leftarrow \frac{C(p,k)}{k!} = \boxed{S(p,k)}$$

$$\boxed{S'(p,k) = k\, S'(p-1,k) + S'(p-1,k-1)}$$

$$S'(p,0) = 1 \quad p = 0$$
$$= 0 \quad p \neq 0$$

$$S'(p,0) = \frac{C(p,0)}{0!}$$

$$\boxed{\begin{array}{l} S'(0,0)=1 \\ S'(p,0)=0 \\ \qquad\qquad p>0 \end{array}}$$

$$n^0 = 1$$

$$n^p = \boxed{\frac{C(p,0)}{0!} n^0} + \underbrace{\frac{C(p,1)}{1!} n^1 + \cdots}$$

$$\underbrace{S'(p,p)} = \frac{C(p,p)}{p!} = 1$$

$$n^p = \underbrace{\frac{C(p,0)}{0!} n^0 + \cdots\cdots + }\ \boxed{\boxed{\frac{C(p,p)}{p!}} n^p}$$

$$\underline{n(n-1)\cdots(n-p+1)}$$

$$n^{p-1} = \sum_{k=0}^{p-1} S(p-1,k)\, n^{\underline{k}}$$

$$n^{p} = \sum_{k=0}^{p} S(p,k)\, n^{\underline{k}}$$

$$n^{p} = n \cdot n^{p-1} = \sum_{k=0}^{p-1} S(p-1,k)\, \overset{n}{\uparrow}\, n^{\underline{k}}$$

$$= \sum_{k=0}^{p-1} S(p-1,k)(n\underbrace{-k}+k)\, n^{\underline{k}}$$

$$= \sum_{k=0}^{p-1} S(p-1,k)\,\underbrace{(n-k)}\, n^{\underline{k}} + \sum_{k=0}^{p-1} S(p-1,k)\, \underbrace{k}\, n^{\underline{k}}$$

$$= \sum_{k=0}^{p-1} S(p-1,k)\, n^{\underline{k+1}} + \sum_{k=1}^{p-1} S(p-1,k)\, k\, n^{\underline{k}}$$

$$= \sum_{k=1}^{p} S(p-1,k-1)\, n^{\underline{k}} + \sum_{k=1}^{p-1} S(p-1,k)\, k\, n^{\underline{k}}$$

$$n^p = S(p-1,\, p-1)\, n^p + \sum_{k=1}^{p-1} \Big[S(p-1,\, k-1) + k \cdot S(p-1,\, k) \Big] n^k$$

$$\sum_{k=0}^{p} S(p,k)\, \underbrace{n^k}$$

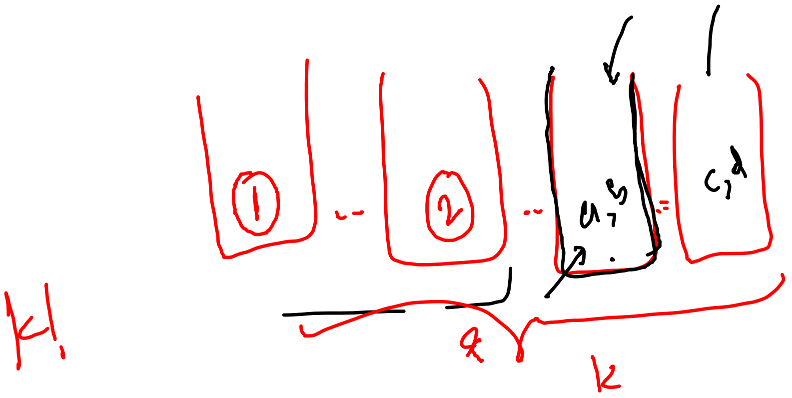
$$\boxed{1 \leq k \leq p-1 \Rightarrow S(p,k) = S(p-1,\, k-1) + k \cdot S(p-1,\, k)}$$

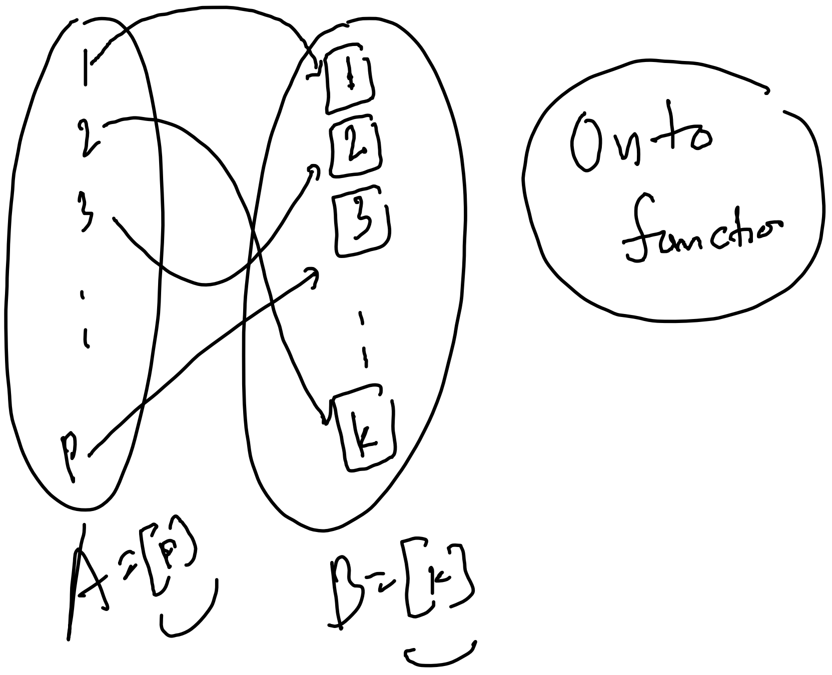
$$\boxed{S(p,k) = \frac{C(p,k)}{k!}}$$

$$\underbrace{n^p} = \sum_{k=0}^{p} \frac{C(p,k)}{k!}\, \underbrace{n^k}$$

$\boxed{C(p,k)} = k!\,(S(p,k))$

[p] [1,2,--p]

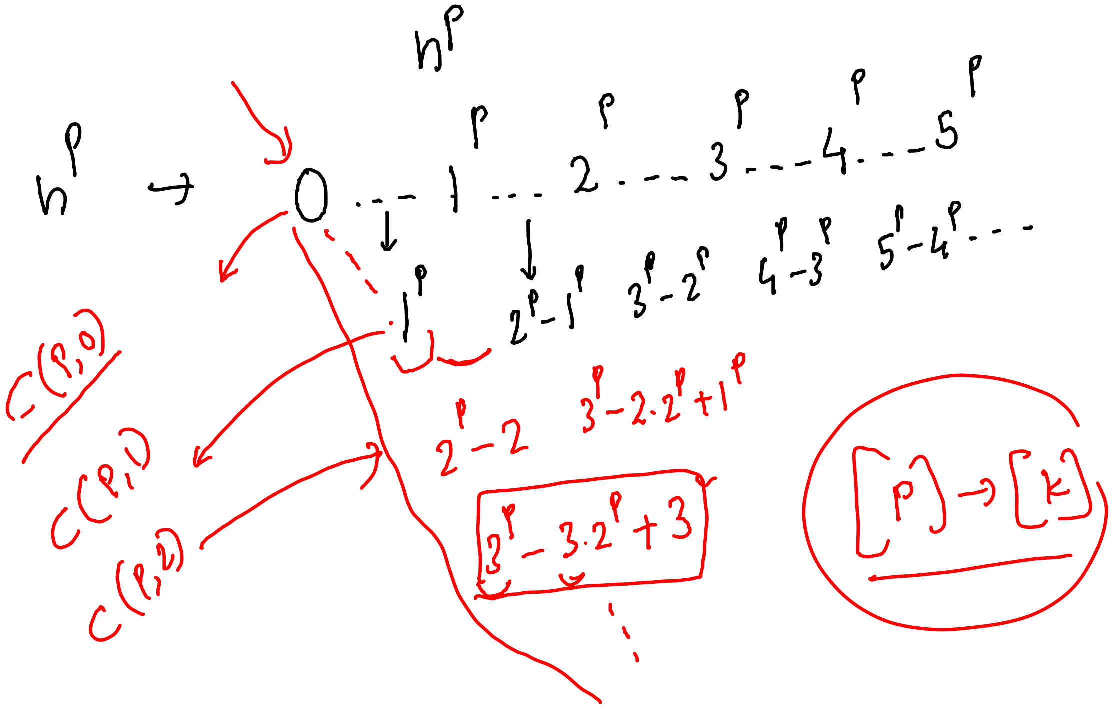




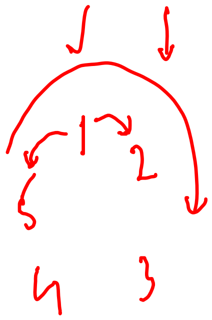
$$\underbrace{k^p - \binom{k}{1}\underbrace{(k-1)}^p + \binom{k}{2}(k-2)^p + \cdots}$$

• a

• b

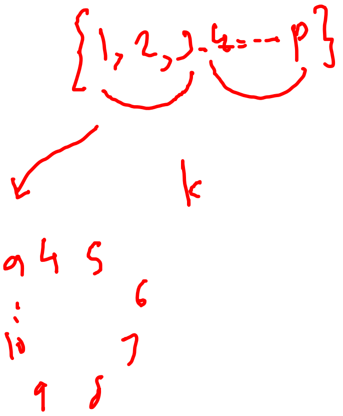


Sterling numbers of the first kind



$(n-1)!$

$s(p,k)$



$s(p,1)$ $(p-1)!$

$$s(p,p) = \boxed{S(p,p)}$$

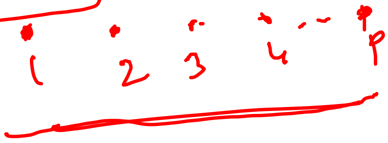
$s(p,k)$

$$\boxed{s(p,k) \geqslant S(p,k)}$$

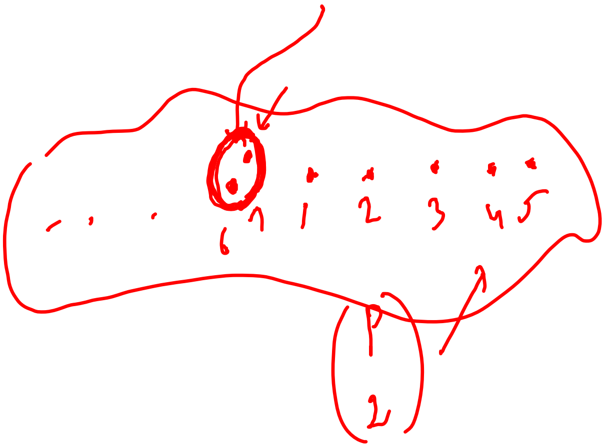
$$\downarrow$$

$$s(p,k) = S(p,k)$$

$k=p$ $$\boxed{s(p,p) = S(p,p) = 1}$$



$$s(p,p-1) = S(p,p-1) = \binom{p}{2}$$



$S(4,2)$ $\{1,2,3,4\}$

$\{1\}$ $\{2,3,4\}$

$\{2\}$ $\{1,3,4\}$

$\{3\}$ $\{1,2,4\}$

$\{4\}$ $\{1,2,3\}$

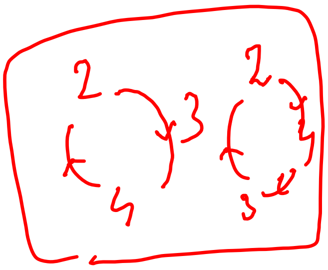
$\{1,2\}$ $\{3,4\}$

$\{1,3\}$ $\{2,4\}$

$\{1,4\}$ $\{3,2\}$

$8+3$
$=11$

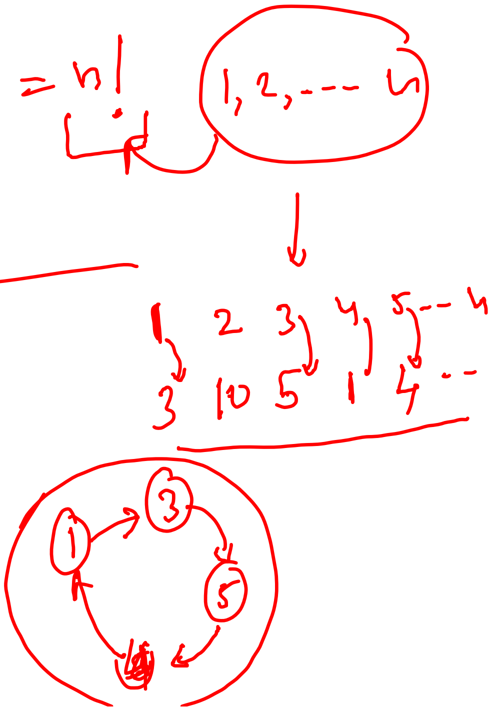
$\{1\}$



$s(4,2)=11$ ✓

$S(4,2)=6$ =

$$\sum_{k=6}^{n} s(n,k) = n!$$



$$n^p = \sum_{k=0}^{p} n^k$$

$S(p,k$

$$n^p = \sum_{k=0}^{p} (-1)^{p-k} s([illegible]) n^k$$